Input-to-state stabilization of linear systems under data-rate constraints

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Abstract

We study feedback stabilization of continuous-time linear systems under finite data-rate constraints in the presence of unknown disturbances. A communication and control strategy based on sampled and quantized state measurements is proposed, where the quantization range is dynamically adjusted using reachable-set propagation and disturbance estimates derived from quantization parameters. The strategy alternates between stabilizing and searching stages to handle escapes from the quantization range and employs an additional quantization symbol to ensure robustness near the equilibrium. It guarantees input-to-state stability (ISS), improving upon existing results that yield only practical ISS or lack explicit data-rate conditions. Simulation results illustrate the effectiveness of the strategy.

1 Introduction

Feedback control under data-rate constraints has been an active research area for decades, as surveyed in [1–3]. Such constraints arise naturally in networked control systems due to communication costs, bandwidth limitations, and security considerations. Beyond these practical motivations, a fundamental question is how much information is required to achieve a given control objective.

In this work, a finite data transmission rate is achieved by generating the control input from sampled and quantized state measurements taking values in a finite set, which is a standard modeling framework. Early developments for linear systems include [4–10]. A central mechanism is dynamic quantization, in which the quantizer range is enlarged to recover the state and then contracted to improve precision near the equilibrium. This approach has been extended to nonlinear systems [11–13] and later to switched systems [14–16].

We consider feedback stabilization under data-rate constraints in the presence of unknown disturbances. In [7, 10], disturbances are assumed to be bounded, and asymptotic stabilization is achieved with minimum data rates. The problem becomes significantly more challenging without such bounds, as disturbances may drive the state outside the quantization range after capture. In this setting, [17] established input-to-state stability (ISS) [18] based on a dynamic quantization scheme with fixed center and alternating "zooming-out" and "zooming-in" stages. Similar ISS properties were achieved in [19] with improved data-rate efficiency, using a moving-center quantizer with escape detection stages and resets. More recently, [16] proposed an adaptive disturbance estimation scheme achieving practical ISS for switched linear systems.

In this paper, we propose a new approach to feedback stabilization under data-rate constraints with completely unknown disturbances. Building on the disturbance estimation idea in [16], we develop a

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communication and control strategy that achieves ISS for linear systems. The key idea is to incorporate a disturbance estimate based on quantization parameters instead of constants, so that the state can escape the quantization range only when the disturbance is sufficiently large relative to both the quantization radius and the distance from its center to the equilibrium. The proposed strategy uses an additional quantization symbol to handle states near the equilibrium and ensure ISS. In contrast to existing approaches [17, 19], our method provides an explicit bound on the admissible data rate and does not require quantizer resets or dedicated escape-detection stages.

The remainder of the paper is organized as follows. Section 2 introduces the system model, information structure, and basic assumptions. Our main result is presented in Section 3. Section 4 explains the communication and control strategy, with reachable-set approximations developed in Section 5. Section 6 provides the stability analysis with major steps summarized as technical lemmas. A simulation example is given in Section 7.

Notations: The ∞-norm of a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ is denoted by $|v| := |v|_\infty = \max_{1\le i\le n} |v_i|$. The induced ∞-norm of a matrix $M = [a_{ij}] \in \mathbb{R}^{n\times n}$ is denoted by $\|M\| := \|M\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$. The largest and smallest eigenvalues of a symmetric matrix $M \in \mathbb{R}^{n\times n}$ are denoted by $\overline{\lambda}(M)$ and $\underline{\lambda}(M)$, respectively. A continuous function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class $\mathcal{K}_\infty$, denoted by $\gamma \in \mathcal{K}_\infty$, if it is strictly increasing, unbounded, and satisfies $\gamma(0) = 0$. The left limit of a piecewise continuous function $z(\cdot)$ at t is denoted by $z(t^-) := \lim_{s\nearrow t} z(s)$.

2 Problem formulation

2.1 System definition

Consider a continuous-time linear system

$$\dot{x} = Ax + Bu + Dd, \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, and $d \in \mathbb{R}^{n_d}$ is an unknown disturbance. The disturbance $d(\cdot)$ is assumed to be Lebesgue measurable and locally essentially bounded. The essential supremum ∞-norm of $d(\cdot)$ over an interval J is denoted by $\|d\|_J := \operatorname{ess\,sup}_{t\in J} |d(t)|$; the subscript is omitted when $J = \mathbb{R}_{\ge 0}$.

Our first basic assumption is that the system (1) is stabilizable.

Assumption 1 (Stabilizability)**.** The pair (A, B) is stabilizable, that is, there exists a state feedback gain matrix K such that $A + BK$ is Hurwitz (all eigenvalues have negative real parts).

In what follows, we assume that such a matrix K has been selected and fixed.

2.2 Information structure

We seek to generate a stabilizing control input $u(\cdot)$ based on limited information about the state $x(\cdot)$. As shown in Fig. 1, the feedback loop includes a sensor with an encoder and a controller with a decoder. The sensor samples the state at times $t_k = k\tau_s$, $k \in \mathbb{Z}_{\ge 0}$, where $\tau_s > 0$ is a fixed *sampling period*. Each sample $x(t_k)$ is encoded as an integer $i_k \in \{0, 1, \dots, N^{n_x} + 1\}$, where $N > 0$ is a fixed integer, and transmitted to

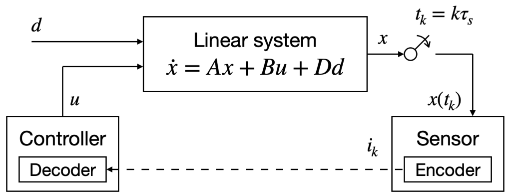

Figure 1: Information structure.

the controller. The resulting data transmission rate is given by

$$\frac{\log_2(N^{n_x} + 2)}{\tau_s}$$

bits per unit time. This information structure enables a separation of sensing and control tasks, as the controller does not require access to the exact state. Details of the communication and control strategy are provided in Section 4.

Our second basic assumption imposes a lower bound on the admissible data rate.

Assumption 2 (Data rate)**.** The sampling period τ_s and the integer N satisfy

$$\Lambda := \|e^{A\tau_s}\| < N. \tag{2}$$

The inequality in (2) can be viewed as a data-rate bound, as it requires the integer N to be sufficiently large relative to the sampling period τ_s. Similar data-rate bounds have appeared in [7, 10, 11] for stabilizing linear systems and in [14, 16] for stabilizing switched linear systems. The relationship among these bounds are discussed in [19, Sec. V] and [14, Sec. 2.2].

3 Main result

The control objective is to stabilize the system defined in Section 2.1 under the information constraint described in Section 2.2 in a robust sense. Specifically, we aim to achieve *input-to-state stability (ISS)*, originally introduced in [18]. The theorem below adopts a characterization of ISS from [20]; see also [21, Sec. 10.4]

Theorem 1. *Consider the linear system* (1)*. Suppose that Assumptions 1 and 2 hold. Then there exists a communication and control strategy that renders the closed-loop system input-to-state stable, that is, the following holds: there exist functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for any initial condition $x_0 \in \mathbb{R}^{n_x}$ and any disturbance $d(\cdot)$, the solution satisfies*

$$|x(t)| \leq \gamma_1(|x_0|) + \gamma_2(\|d\|) \qquad \forall\, t \geq 0 \tag{3}$$

and

$$\limsup_{t\to\infty} |x(t)| \leq \gamma_3\Big(\limsup_{t\to\infty} |d(t)|\Big). \tag{4}$$

The communication and control strategy is presented in Section 4. The $\mathcal{K}_\infty$ gain functions γ_1, γ_2, and γ_3 are given in (44) and (45) in the proof of Theorem 1 in Section 6.3.

4 Communication and control strategy

In this section, we describe the communication and control strategy, assuming that suitable approximations of the reachable sets of the state are available at all sampling times (these approximations are constructed in Section 5).

The initial state $x(0) = x_0$ is unknown. At $t_0 = 0$, both the sensor and the controller are initialized with $x_0^* = 0$ and a design parameter $E_0 > 0$. At each sampling time t_k, the sensor checks whether

$$|x(t_k) - x_k^*| \leq E_k, \tag{5}$$

that is, whether the state $x(t_k)$ lies in the hypercube

$$\mathcal{R}_k := \{v \in \mathbb{R}^{n_x} : |v - x_k^*| \leq E_k\}.$$

The set $\mathcal{R}_k$ approximates the reachable set at t_k and serves as the quantization range. If (5) holds, then the state is *visible*, and the system is in a *stabilizing stage* (Section 4.1); otherwise, the state is *lost*, and the system is in a *searching stage* (Section 4.2).

If the state is visible at t_k, then the system remains in a stabilizing stage until the first $t_j > t_k$ such that $x(t_j) \notin \mathcal{R}_j$, at which point the state *escapes*. Conversely, if the state is lost at t_k, then the system remains in a searching stage until the first $t_i > t_k$ such that $x(t_i) \in \mathcal{R}_i$, at which point the state is *(re)captured*. Due to the unknown disturbance, the system may alternate between stabilizing and searching stages finitely or infinitely many times.

4.1 Stabilizing stages

At each t_k in a stabilizing stage, the sensor first checks whether

$$|x(t_k)| \leq \frac{E_k}{N}. \tag{6}$$

If so, it transmits $i_k = 1$ to the controller. Otherwise, it partitions $\mathcal{R}_k$ into N^{n_x} equal hypercubic cells (with N per dimension), assigns each cell a unique index from $\{2, \dots, N^{n_x} + 1\}$, and transmits the index i_k of the cell containing $x(t_k)$. Each such cell has radius E_k/N.

Upon receiving $i_k \in \{1, \dots, N^{n_x} + 1\}$, the controller infers that (5) holds and reconstructs the corresponding cell center c_k using the same indexing protocol. For $i_k \geq 2$, we have

$$|x(t_k) - c_k| \leq \frac{E_k}{N} \tag{7}$$

and

$$|c_k - x_k^*| \leq \frac{N-1}{N} E_k. \tag{8}$$

For $i_k = 1$, we have $c_k = 0$ and (7) still holds.

The controller then applies the control input

$$u(t) = K\hat{x}(t), \qquad t \in [t_k, t_{k+1}),$$

where K is the stabilizing gain matrix from Assumption 1, and $\hat{x}$ evolves according to the auxiliary system

$$\dot{\hat{x}} = A\hat{x} + Bu, \tag{9}$$

with the boundary condition

$$\hat{x}(t_k) = c_k. \tag{10}$$

Hence $\hat{x}$ is reset to c_k at each t_k in a stabilizing stage, and is in general only right-continuous.

Finally, both the sensor and the controller compute

$$\begin{aligned} x^*_{k+1} &:= F(c_k), \\ E_{k+1} &:= G(E_k, x^*_k) \end{aligned} \tag{11}$$

without further communication. The functions F and G, derived in Section 5.1, are designed to satisfy two properties. First, they ensure

$$|x(t_{k+1}) - x^*_{k+1}| \leq E_{k+1} \tag{12}$$

whenever $\|d\|_{[t_k,t_{k+1}]} = 0$. Second, if the state escapes at t_{k+1} (i.e., (12) does not hold), they provide bounds on $|x(t_{k+1})|$ and E_{k+1} in terms of $\|d\|_{[t_k,t_{k+1}]}$.

4.2 Searching stages

At each t_k in a searching stage, the sensor transmits the "overflow symbol" $i_k = 0$. Upon receiving $i_k = 0$, the controller infers that the state is lost and then applies the control input $u \equiv 0$ on $[t_k, t_{k+1})$. Finally, both the sensor and the controller compute

$$\begin{aligned} x^*_{k+1} &:= \hat{F}(x^*_k), \\ E_{k+1} &:= \hat{G}((1+\varepsilon)E_k, \delta) \end{aligned} \tag{13}$$

using design parameters $\varepsilon, \delta > 0$, without further communication. The functions $\hat{F}$ and $\hat{G}$, derived in Section 5.2, are designed so that

$$|x(t_{k+1}) - x^*_{k+1}| \leq \hat{G}(|x(t_k) - x^*_k|, \|d\|_{[t_k,t_{k+1}]}). \tag{14}$$

Hence the factor $1+\varepsilon$ in (13) ensures that the growth rate of E_k dominates that of $|x(t_k) - x^*_k|$.

At an escape time t_j, the computation of E_{j+1} is adjusted: the first argument of $\hat{G}$ is modified to enable comparison between δ and $\|d\|_{[t_{j-1},t_{j+1}]}$; see Section 5.2 for details.

5 Approximation of reachable sets

In this section, we derive recursive formulas for propagating reachable-set approximations used in the communication and control strategy.

5.1 Stabilizing stages

Consider a sampling time t_k in a stabilizing stage, so that (5) holds. Define the error $e := x - \hat{x}$. Combining (1) and (9) yields

$$\dot{e} = Ae + Dd, \qquad |e(t_k)| = |x(t_k) - c_k| \leq \frac{E_k}{N} \tag{15}$$

on $[t_k, t_{k+1})$, where the boundary condition follows from (7) and (10). Hence

$$\begin{aligned} |e(t_{k+1}^-)| &= \left| e^{A\tau_s} e(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} Dd(\tau) \, \mathrm{d}\tau \right| \\ &\leq \|e^{A\tau_s}\| |e(t_k)| + \left(\int_0^{\tau_s} \|e^{As} D\| \, \mathrm{d}s \right) \|d\|_{[t_k, t_{k+1}]} \\ &\leq \frac{\Lambda}{N} E_k + \Phi \|d\|_{[t_k, t_{k+1}]}, \end{aligned} \tag{16}$$

where $\Lambda = \|e^{A\tau_s}\|$ as in (2) and

$$\Phi := \int_0^{\tau_s} \|e^{As} D\| \, \mathrm{d}s. \tag{17}$$

We define the propagation functions as

$$x_{k+1}^* = F(c_k) := \hat{x}(t_{k+1}^-) = Sc_k, \tag{18}$$

where $S := e^{(A+BK)\tau_s}$, and

$$E_{k+1} = G(E_k, x_k^*) := \frac{\Lambda}{N} E_k + \sqrt{\phi V_k} \tag{19}$$

with

$$V_k := V(x_k^*, E_k) := (x_k^*)^\top P x_k^* + \rho E_k^2, \tag{20}$$

where $\phi, \rho > 0$ are design parameters specified below.

Because $A + BK$ is Hurwitz, there exist positive definite symmetric matrices $P, Q \in \mathbb{R}^{n_x \times n_x}$ such that

$$S^\top PS - P = -Q < 0. \tag{21}$$

Define

$$\chi := \frac{2n_x^2 \|S^\top PS\|^2}{\underline{\lambda}(Q)} + n_x \|S^\top PS\|. \tag{22}$$

We select design parameters $\psi, \rho, \phi > 0$ sequentially. As $\Lambda < N$ in Assumption 2, there exists a sufficiently small $\psi > 0$ such that

$$(1 + \psi) \frac{\Lambda^2}{N^2} < 1, \tag{23}$$

and then a sufficiently large $\rho > 0$ such that

$$\frac{(N-1)^2}{N^2} \frac{\chi}{\rho} + (1 + \psi) \frac{\Lambda^2}{N^2} < 1. \tag{24}$$

Finally, there exists a sufficiently small $\phi > 0$ such that

$$\begin{aligned}\nu := \max\left\{1 - \frac{\underline{\lambda}(Q)}{2\,\overline{\lambda}(P)}, \frac{(N-1)^2}{N^2}\frac{\chi}{\rho} + (1+\psi)\frac{\Lambda^2}{N^2}\right\}\\ + \left(1 + \frac{1}{\psi}\right)\phi\rho\end{aligned} \tag{25}$$

satisfies $\nu < 1$.

5.2 Searching stages

Consider a sampling time t_k in a searching stage. Combining (1) and (9) with $u \equiv 0$ and $\hat{x}(t_k) = x_k^*$ yields

$$\dot{e} = Ae + Dd, \qquad |e(t_k)| = |x(t_k) - x_k^*| \tag{26}$$

on $[t_k, t_{k+1})$. Hence

$$\begin{aligned}|e(t_{k+1}^-)| &\le \|e^{A\tau_s}\||e(t_k)| + \left(\int_0^{\tau_s} \|e^{As}D\|\,\mathrm{d}s\right)\|d\|_{[t_k,t_{k+1}]}\\ &\le \Lambda|e(t_k)| + \Phi\|d\|_{[t_k,t_{k+1}]}\end{aligned} \tag{27}$$

where Λ and Φ are defined in (2) and (17), respectively.

We define the propagation functions as

$$x_{k+1}^* = \hat{F}(x_k^*) := \hat{x}(t_{k+1}^-) = \hat{S}x_k^*, \tag{28}$$

where $\hat{S} := e^{A\tau_s}$, and, to dominate the growth rate of $|e(t_k)|$,

$$E_{k+1} = \hat{G}((1+\varepsilon)E_k, \delta) := (1+\varepsilon)\Lambda E_k + \Phi\delta, \tag{29}$$

where $\varepsilon, \delta > 0$ are design parameters.

At an escape time t_j, the computation of E_{j+1} is adjusted to enable comparison between δ and $\|d\|_{[t_{j-1},t_{j+1}]}$. Instead of using E_j from (19) in the first argument, we let

$$E_{j+1} = \hat{G}((1+\varepsilon)\hat{E}_j, \delta) \tag{30}$$

where

$$\hat{E}_j := \frac{\Lambda}{N}E_{j-1} + \Phi\delta.$$

In Section 6.2, we show that the state is(re)captured in finite time under essentially bounded disturbances. The comparison between δ and $\|d\|_{[t_{j-1},t_{j+1}]}$ is established in the proof of Lemma 8.

6 Stability analysis

In this section, we establish Theorem 1 based on the communication and control strategy described in Section 4. We first present key steps of the proof as technical lemmas, followed by the proof of Theorem 1 in Section 6.3.

Throughout the analysis, we assume that the essential supremum norm $\|d\|$ is finite, since otherwise the bounds (3) and (4) hold trivially. We also assume that $\Lambda > 1$.[1]

[1] Since $\Lambda = \|e^{A\tau_s}\| \ge 1$, equality holds only if all eigenvalues of A have nonpositive real parts. The case $\Lambda = 1$ can be handled by replacing Λ with $\max\{\|e^{A\tau_s}\|,\ 1 + \epsilon_\Lambda\}$ for any $\epsilon_\Lambda > 0$.

6.1 Stabilizing stages

We first establish exponential decay of $V_k := V(x_k^*, E_k)$ defined in (20) during stabilizing stages.

Lemma 1. *For any sampling time t_k in a stabilizing stage,*

$$V_{k+1} \le \nu V_k, \tag{31}$$

where $\nu \in (0, 1)$ is defined in (25).

Proof. See Appendix A.1. $\square$

We next relate V_k to $|x(t_k)|$, $|x(t_{k+1})|$, and E_k.

Lemma 2. *There exist constants $C_1, C_2, C_3 > 0$ such that for any sampling time t_k in a stabilizing stage,*

$$\sqrt{V_k} \le C_1(|x(t_k)| + E_k), \tag{32}$$

$$|x(t_k)| \le C_2\sqrt{V_k}, \tag{33}$$

$$|x(t_{k+1})| \le C_3\sqrt{V_k} + \Phi\|d\|_{[t_k,t_{k+1}]}, \tag{34}$$

Proof. See Appendix A.2. $\square$

The next two lemmas provide an exponentially decaying state bound and a state bound with suitable monotonicity and $\mathcal{K}_\infty$ properties.

Lemma 3. *There exist constants $C, \lambda > 0$ such that for any two sampling times $t_k > t_l$ from the same stabilizing stage,*

$$|x(t_k)| \le Ce^{-\lambda(k-l)}(|x(t_l)| + E_l) + \Phi\|d\|_{[t_{k-1},t_k]}. \tag{35}$$

Proof. See Appendix A.3. $\square$

Lemma 4. *There exist continuous functions $\chi^x, \chi^d : \mathbb{R}_{>0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that for each fixed $s > 0$, $\chi^x(\cdot, s)$ and $\chi^d(\cdot, s)$ are nondecreasing, for each fixed $E > 0$, $\chi^x(E, \cdot), \chi^d(E, \cdot) \in \mathcal{K}_\infty$, and for any two sampling times $t_k \ge t_l$ from the same stabilizing stage,*

$$|x(t_k)| \le \chi^x(E_l, |x(t_l)|) + \chi^d(E_l, \|d\|_{[t_l,t_k]}). \tag{36}$$

Proof. See Appendix A.4. $\square$

Finally, we bound the state and quantization radius at escape times in terms of the disturbance over the preceding sampling interval, independently of the initial state.

Lemma 5. *There exists a constant $\Gamma > 0$ such that for any sampling time t_j at which the state escapes,*

$$|x(t_j)| \le \Gamma\|d\|_{[t_{j-1},t_j]}, \qquad E_{j-1} \le \Gamma\|d\|_{[t_{j-1},t_j]}. \tag{37}$$

Proof. See Appendix A.5. $\square$

6.2 Searching stages

We first show that if the state is lost at $t_0 = 0$, then it is guaranteed to be captured in finite time.

Lemma 6. *If the state is lost at $t_0 = 0$, then it is captured at some sampling time t_{i_0} satisfying*

$$i_0 \leq \max\left\{\eta_x\left(\frac{|x_0|}{E_0}\right),\, \eta_d\left(\frac{\|d\|_{[0,t_{i_0}]}}{\delta}\right)\right\}, \tag{38}$$

where

$$\eta_x(s) := \begin{cases} \lceil \log_{1+\varepsilon} s \rceil, & s > 1, \\ 0, & 0 \leq s \leq 1, \end{cases}$$

$$\eta_d(s) := \begin{cases} \lceil \log_{1+\varepsilon} (r_\varepsilon s) \rceil, & s > 1, \\ 0, & 0 \leq s \leq 1, \end{cases}$$

with

$$r_\varepsilon := \frac{\hat{\Lambda} - 1}{\Lambda - 1}, \qquad \hat{\Lambda} := (1+\varepsilon)\Lambda.$$

Proof. See Appendix A.6. □

The next lemma provides bounds on the state and the quantization radius at the first capture.

Lemma 7. *There exist functions $\hat{\gamma}_0^x, \hat{\gamma}_0^d \in \mathcal{K}_\infty$ such that if the state is first captured at t_{i_0}, then for any sampling time $t_k \leq t_{i_0}$,*

$$|x(t_k)| \leq \hat{\gamma}_0^x(|x_0|) + \hat{\gamma}_0^d(\|d\|_{[0,t_{i_0}]}). \tag{39}$$

Moreover, there exists a continuous function $\hat{\chi}_0^E : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that for each fixed $E, s > 0$, $\hat{\chi}_0^E(E, s, \cdot)$ and $\hat{\chi}_0^E(E, \cdot, s)$ are nondecreasing, and

$$E_{i_0} \leq \hat{\chi}_0^E(E_0, |x_0|, \|d\|_{[0,t_{i_0}]}). \tag{40}$$

Proof. See Appendix A.7. □

We now show that if the state escapes, then it is guaranteed to be recaptured in finite time.

Lemma 8. *If the state escapes at $t_j > 0$, then it is recaptured at some sampling time $t_i > t_j$ satisfying*

$$i \leq j + \max\left\{\eta_d\left(\frac{\|d\|_{[t_{j-1},t_i]}}{\delta}\right),\, 1\right\}, \tag{41}$$

where η_d is defined in Lemma 6.

Proof. See Appendix A.8. □

The final lemma provides bounds on the state and the quantization radius at recapture.

Lemma 9. *There exist $\hat{\gamma}^x, \hat{\gamma}^d \in \mathcal{K}_\infty$ such that if the state escapes at t_j and is recaptured at t_i, then for any sampling time t_k between t_j and t_i,*

$$|x(t_k)| \leq \hat{\gamma}^x(|x(t_j)|) + \hat{\gamma}^d(\|d\|_{[t_{j-1},t_i]}). \tag{42}$$

Moreover, there exists a continuous function $\hat{\chi}^E : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that for each fixed $E > 0$, $\hat{\chi}^E(E, \cdot)$ is nondecreasing, and

$$E_i \leq \hat{\chi}^E(E_{j-1}, \|d\|_{[t_{j-1},t_i]}). \tag{43}$$

Proof. See Appendix A.9. □

6.3 Proof of Theorem 1

We index the alternating searching and stabilizing stages as follows. Let $0 \leq i_0 < j_1 < i_1 < \cdots$ be such that the state is first captured at t_{i_0}, escapes at t_{j_l}, and is recaptured at t_{i_l} for $l \geq 1$. By Lemma 6, we have $i_0 < \infty$. If the state never escapes after some t_{i_l}, we set $j_{l+1} = \infty$. By Lemma 8, if $j_l < \infty$ then $i_l < \infty$.

First, we establish the bound (3) at sampling times by constructing the functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$.

First searching stage $[0, t_{i_0})$ Assume without loss of generality that the state is lost at $t_0 = 0$; otherwise, set $i_0 = 0$. By Lemma 7, for all $t_k \leq t_{i_0}$, the bound (39) holds.

First stabilizing stage $[t_{i_0}, t_{j_1})$ At t_{i_0}, Lemma 7 yields

$$|x(t_{i_0})| \leq \hat{\gamma}_0^x(|x_0|) + \hat{\gamma}_0^d(\|d\|_{[0,t_{i_0}]}),$$

and

$$E_{i_0} \leq \hat{\chi}_0^E(E_0, |x_0|, \|d\|_{[0,t_{i_0}]}).$$

From Lemma 4, for any $t_{i_0} \leq t_k < t_{j_1}$,

$$\begin{aligned}|x(t_k)| &\leq \chi^x(E_{i_0}, |x(t_{i_0})|) + \chi^d(E_{i_0}, \|d\|_{[t_{i_0},t_k]}) \\ &\leq \chi^x\Big(\hat{\chi}_0^E(E_0, |x_0|, \|d\|_{[0,t_{i_0}]}), \hat{\gamma}_0^x(|x_0|) + \hat{\gamma}_0^d(\|d\|_{[0,t_{i_0}]})\Big) \\ &\quad + \chi^d\Big(\hat{\chi}_0^E(E_0, |x_0|, \|d\|_{[0,t_{i_0}]}), \|d\|_{[t_{i_0},t_k]}\Big).\end{aligned}$$

Using the above bounds, a standard comparison-function argument yields continuous functions $\bar{\chi}_1, \bar{\chi}_2 : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for each fixed $E > 0$, $\bar{\chi}_1(E, \cdot), \bar{\chi}_2(E, \cdot) \in \mathcal{K}_\infty$, and

$$\begin{aligned}&\bar{\chi}_1(E_0, |x_0|) + \bar{\chi}_2(E_0, \|d\|_{[0,t_k]}) \\ &\quad \geq \chi^x\Big(\hat{\chi}_0^E(E_0, |x_0|, \|d\|_{[0,t_k]}), \hat{\gamma}_0^x(|x_0|) + \hat{\gamma}_0^d(\|d\|_{[0,t_k]})\Big) \\ &\qquad + \chi^d\Big(\hat{\chi}_0^E(E_0, |x_0|, \|d\|_{[0,t_k]}), \|d\|_{[0,t_k]}\Big),\end{aligned}$$

which implies

$$|x(t_k)| \leq \bar{\chi}_1(E_0, |x_0|) + \bar{\chi}_2(E_0, \|d\|_{[0,t_k]}).$$

If $t_{j_1} = \infty$, then the proof of (3) is complete. Otherwise, consider an arbitrary escape time $t_{j_l} < \infty$.

Searching stage $[t_{j_l}, t_{i_l})$ At t_{j_l}, Lemma 5 yields

$$|x(t_{j_l})| \leq \Gamma \|d\|_{[t_{j_l-1}, t_{j_l}]}, \qquad E_{j_l-1} \leq \Gamma \|d\|_{[t_{j_l-1}, t_{j_l}]}.$$

From Lemma 9, for any $t_{j_l} \leq t_k \leq t_{i_l}$,

$$\begin{aligned} |x(t_k)| &\leq \hat{\gamma}^x(|x(t_{j_l})|) + \hat{\gamma}^d(\|d\|_{[t_{j_l-1}, t_{i_l}]}) \\ &\leq \hat{\gamma}^x(\Gamma \|d\|_{[t_{j_l-1}, t_{j_l}]}) + \hat{\gamma}^d(\|d\|_{[t_{j_l-1}, t_{i_l}]}) \\ &\leq \hat{\gamma}(\|d\|_{[t_{j_l-1}, t_{i_l}]}), \end{aligned}$$

where $\hat{\gamma}(s) := \hat{\gamma}^x(\Gamma s) + \hat{\gamma}^d(s) \in \mathcal{K}_\infty$.

Stabilizing stage $[t_{i_l}, t_{j_{l+1}})$ At t_{i_l},

$$|x(t_{i_l})| \leq \hat{\gamma}(\|d\|_{[t_{j_l-1}, t_{i_l}]}),$$

and Lemma 9 yields

$$\begin{aligned} E_{i_l} &\leq \hat{\chi}^E(E_{j_l-1}, \|d\|_{[t_{j_l-1}, t_{i_l}]}) \\ &\leq \hat{\chi}^E(\Gamma \|d\|_{[t_{j_l-1}, t_{j_l}]}, \|d\|_{[t_{j_l-1}, t_{i_l}]}). \end{aligned}$$

From Lemma 4, for any $t_{i_l} \leq t_k < t_{j_{l+1}}$,

$$\begin{aligned} |x(t_k)| &\leq \chi^x(E_{i_l}, |x(t_{i_l})|) + \chi^d(E_{i_l}, \|d\|_{[t_{i_l}, t_k]}) \\ &\leq \chi^x\Big(\hat{\chi}^E(\Gamma \|d\|_{[t_{j_l-1}, t_{j_l}]}, \|d\|_{[t_{j_l-1}, t_{i_l}]}), \hat{\gamma}(\|d\|_{[t_{j_l-1}, t_{i_l}]})\Big) \\ &\quad + \chi^d\Big(\hat{\chi}^E(\Gamma \|d\|_{[t_{j_l-1}, t_{j_l}]}, \|d\|_{[t_{j_l-1}, t_{i_l}]}), \|d\|_{[t_{i_l}, t_k]}\Big) \\ &\leq \bar{\gamma}(\|d\|_{[t_{j_l-1}, t_k]}), \end{aligned}$$

where $\bar{\gamma}(s) := \chi^x(\hat{\chi}^E(\Gamma s, s), \hat{\gamma}(s)) + \chi^d(\hat{\chi}^E(\Gamma s, s), s) \in \mathcal{K}_\infty$.

By induction over consecutive searching and stabilizing stages, the bound (3) holds for all sampling times with the $\mathcal{K}_\infty$ functions

$$\begin{aligned} \gamma_1(s) &:= \max\{\hat{\gamma}_0^x(s),\ \bar{\chi}_1(E_0, s)\}, \\ \gamma_2(s) &:= \max\{\hat{\gamma}_0^d(s),\ \bar{\chi}_2(E_0, s),\ \hat{\gamma}(s),\ \bar{\gamma}(s)\}. \end{aligned} \tag{44}$$

Next, we establish the bound (4) at sampling times by constructing the function $\gamma_3 \in \mathcal{K}_\infty$. If only finitely many searching stages occur, then after some time the system remains in a stabilizing stage, and Lemma 3 implies

$$\limsup_{k \to \infty} |x(t_k)| \leq \Phi \limsup_{t \to \infty} |d(t)|.$$

If infinitely many searching stages occur, then the bounds above imply that, after each escape time t_{j_l},

$$\begin{aligned} |x(t_k)| &\leq \hat{\gamma}(\|d\|_{[t_{j_l-1}, t_{i_l}]}) \qquad \forall\, j_l \leq k \leq i_l, \\ |x(t_k)| &\leq \bar{\gamma}(\|d\|_{[t_{j_l-1}, t_k]}) \qquad \forall\, i_l \leq k < j_{l+1}. \end{aligned}$$

Therefore,

$$\limsup_{k\to\infty} |x(t_k)| \le \gamma_3\Big(\limsup_{t\to\infty} |d(t)|\Big)$$

with the $\mathcal{K}_\infty$ function

$$\gamma_3(s) := \max\{\Phi s,\, \hat\gamma(s),\, \bar\gamma(s)\}. \tag{45}$$

The extension of (3) and (4) to all $t \ge 0$ follows from standard arguments. Specifically, for any $t \ge 0$, let k be such that $t \in [t_k, t_{k+1})$. We first consider the case that the system is in a stabilizing stage at t_k and $c_k \ne 0$. From (9) and (15),

$$\begin{aligned}
|x(t)| &\le |\hat x(t)| + |e(t)| \\
&\le \|e^{(A+BK)(t-t_k)}\||c_k| \\
&\quad + \|e^{A(t-t_k)}\||e(t_k)| + \left(\int_0^t \|e^{As}D\|\,\mathrm{d}s\right)\|d\|_{[t_k,t]} \\
&\le \bar\Lambda|c_k| + \frac{\tilde\Lambda}{N}E_k + \Phi\|d\|_{[t_k,t]},
\end{aligned}$$

where

$$\bar\Lambda := \max_{0\le s\le \tau_s} \|e^{(A+BK)s}\|, \qquad \tilde\Lambda := \max_{0\le s\le \tau_s} \|e^{As}\|.$$

By (7),

$$|c_k| \le |x(t_k)| + |x(t_k) - c_k| \le |x(t_k)| + \frac{E_k}{N},$$

and since (6) fails at t_k,

$$|x(t_k)| > \frac{E_k}{N}.$$

Hence

$$|x(t)| \le \tilde H|x(t_k)| + \Phi\|d\|_{[t_k,t]}. \tag{46}$$

with

$$\tilde H := 2\bar\Lambda + \tilde\Lambda.$$

If the system is in a stabilizing stage at t_k and $c_k = 0$, or the system is in a searching stage at t_k, then $u \equiv 0$ on $[t_k, t_{k+1})$, and

$$\begin{aligned}
|x(t)| &\le \|e^{A(t-t_k)}\||x(t_k)| + \left(\int_0^{\tau_s} \|e^{As}D\|\,\mathrm{d}s\right)\|d\|_{[t_k,t]} \\
&\le \tilde\Lambda|x(t_k)| + \Phi\|d\|_{[t_k,t]},
\end{aligned}$$

so (46) still holds.

Since the bound (3) holds for all sampling times with the $\mathcal{K}_\infty$ functions γ_1 and γ_2 defined in (44), by (46), it holds for all $t \ge 0$ with the $\mathcal{K}_\infty$ functions

$$\begin{aligned}
\gamma_1(s) &:= \tilde H \max\{\hat\gamma_0^x(s),\, \bar\chi_1(E_0, s)\}, \\
\gamma_2(s) &:= \tilde H \max\{\hat\gamma_0^d(s),\, \bar\chi_2(E_0, s),\, \hat\gamma(s),\, \bar\gamma(s)\} + \Phi s.
\end{aligned}$$

Since

$$\limsup_{k\to\infty} |x(t_k)| \leq \gamma_3\Big(\limsup_{t\to\infty} |d(t)|\Big)$$

with the $\mathcal{K}_\infty$ functions γ_3 defined in (45), by (46),

$$\begin{aligned}\limsup_{t\to\infty} |x(t)| &\leq \limsup_{k\to\infty} \tilde{H}|x(t_k)| + \Phi\|d\|_{[t_k,t_{k+1}]} \\ &\leq \limsup_{k\to\infty} \tilde{H}|x(t_k)| + \Phi \limsup_{t\to\infty} |d(t)|,\end{aligned}$$

which implies the bound (4) with the $\mathcal{K}_\infty$ function

$$\gamma_3(s) := \tilde{H} \max\{\Phi s,\, \hat{\gamma}(s),\, \bar{\gamma}(s)\} + \Phi s.$$

7 Simulation example

We illustrate the proposed communication and control strategy on the linear system (1) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1.5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ K = \begin{bmatrix} -3.5 & 0 \end{bmatrix},$$

sampling period $\tau_s = 0.1$ s, and integer $N = 5$, which satisfy Assumptions 1 and 2. The design parameters are chosen to satisfy the conditions in Section 5:

$$E_0 = 0.5,\ \varepsilon = 0.2,\ \delta = 0.1,\ \psi = 0.2,\ \rho = 0.1,\ \phi = 0.01.$$

We simulate the system with initial state

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so the state is initially lost. To induce escape and recovery events, disturbance pulses with $d(t) = 1.5$ are applied on $[10.5, 10.7]$ s and $[22.5, 22.7]$ s, and $d(t) = 0$ otherwise.

Figure 2(a) shows the error $|e(t)| = |x(t) - \hat{x}(t)|$ (blue) and quantization radius E_k (red). During stabilizing stages, both $|e(t_k)|$ and E_k at sampling times decay exponentially; $|e(t)|$ may increase between sampling times; E_k are only computed at sampling times and plotted as step functions. Disturbance pulses (onsets indicated by black dashed lines) cause sharp increases in $|e(t)|$, leading to $|e(t_k)| > E_k$ and loss of the state. During searching stages (shaded regions), the growth of E_k dominates that of $|e(t_k)|$, enabling recapture of the state.

Figure 2(b) shows the state $x_1(t)$ (blue) and auxiliary state $\hat{x}_1(t)$ (red). During stabilizing stages, $x_1(t)$ is driven towards zero (it could exhibit temporary increase due to the quantization error); $\hat{x}_1(t)$ decreases continuously between sampling times but exhibits discontinuities at sampling times due to new measurements. During searching stages (shaded regions), $x_1(t)$ and $\hat{x}_1(t)$ both grow (the latter is unclear since the values are small). These behaviors are consistent with the ISS properties established in Theorem 1.

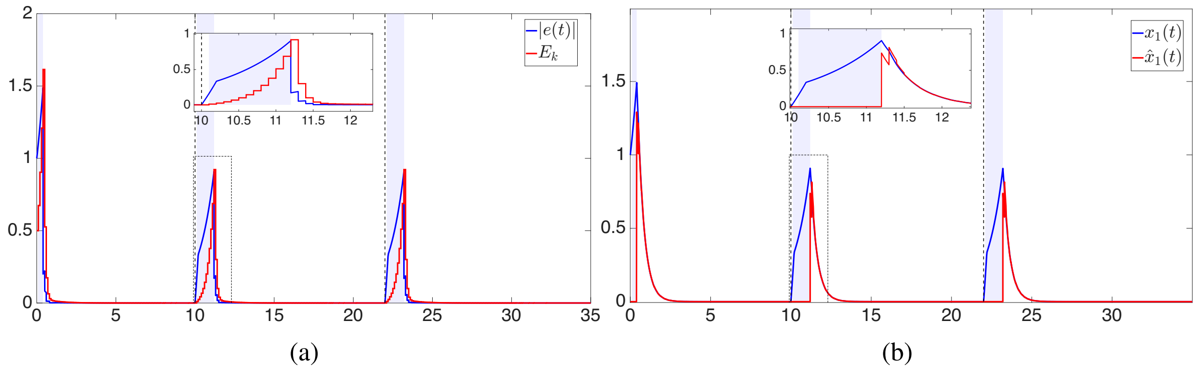

Figure 2: (a) Error $|e(t)| = |x(t) - \hat{x}(t)|$ (blue) and quantization radius E_k (red). (b) State $x_1(t)$ (blue) and auxiliary state $\hat{x}_1(t)$ (red). Black dashed lines indicate disturbances onsets. Shaded regions indicate searching stages. The zoomed-in plots highlight the searching stage and subsequent recapture after the first escape.

A Proofs of technical lemmas

We first recall several basic facts from linear algebra used throughout the proofs. For any $v, w \in \mathbb{R}^n$, the ∞-norm satisfies

$$|v|^2 \le v^\top v, \quad v^\top w \le n|v||w|. \tag{47}$$

For any symmetric matrix $M \in \mathbb{R}^{n\times n}$ and any $v \in \mathbb{R}^n$,

$$\underline{\lambda}(M)\, v^\top v \le v^\top M v \le \overline{\lambda}(M)\, v^\top v, \tag{48}$$

which yields

$$\underline{\lambda}(M)|v|^2 \le v^\top M v \le n\,\overline{\lambda}(M)|v|^2. \tag{49}$$

A.1 Proof of Lemma 1

We first consider the case $c_k \neq 0$. From the propagation function (18),

$$x^*_{k+1} = Sc_k = S(x^*_k + \Delta_k), \qquad \Delta_k := c_k - x^*_k,$$

where

$$|\Delta_k| \le \frac{N-1}{N} E_k$$

by (8). Then

$$\begin{aligned}
&(x^*_{k+1})^\top P x^*_{k+1} \\
&= (x^*_k + \Delta_k)^\top S^\top P S (x^*_k + \Delta_k) \\
&= (x^*_k)^\top S^\top P S x^*_k + 2(x^*_k)^\top S^\top P S \Delta_k + \Delta_k^\top S^\top P S \Delta_k.
\end{aligned}$$

Using the Lyapunov equation (21) and bounds (48) and (49), we obtain

$$
\begin{aligned}
(x_k^*)^\top S^\top P S x_k^* &= (x_k^*)^\top (P - Q) x_k^* \\
&= (x_k^*)^\top P x_k^* - \frac{1}{2}(x_k^*)^\top Q x_k^* - \frac{1}{2}(x_k^*)^\top Q x_k^* \\
&\leq \left(1 - \frac{\underline{\lambda}(Q)}{2\,\overline{\lambda}(P)}\right)(x_k^*)^\top P x_k^* - \frac{1}{2}\,\underline{\lambda}(Q)|x_k^*|^2.
\end{aligned}
$$

Moreover, by (47),

$$
\begin{aligned}
2(x_k^*)^\top S^\top P S \Delta_k &\leq 2n_x |x_k^*| \|S^\top P S\| |\Delta_k|, \\
\Delta_k^\top S^\top P S \Delta_k &\leq n_x \|S^\top P S\| |\Delta_k|^2.
\end{aligned}
$$

Combining the bounds and completing the square yields

$$
\begin{aligned}
&(x_{k+1}^*)^\top P x_{k+1}^* \\
&\leq \left(1 - \frac{\underline{\lambda}(Q)}{2\,\overline{\lambda}(P)}\right)(x_k^*)^\top P x_k^* - \frac{1}{2}\,\underline{\lambda}(Q)|x_k^*|^2 \\
&\quad + 2n_x |x_k^*| \|S^\top P S\| |\Delta_k| + n_x \|S^\top P S\| |\Delta_k|^2 \\
&\leq \left(1 - \frac{\underline{\lambda}(Q)}{2\,\overline{\lambda}(P)}\right)(x_k^*)^\top P x_k^* + \chi |\Delta_k|^2 \\
&\quad - \left(\sqrt{\frac{1}{2}\,\underline{\lambda}(Q)}|x_k^*| - \frac{\sqrt{2} n_x \|S^\top P S\|}{\sqrt{\underline{\lambda}(Q)}}|\Delta_k|\right)^2 \\
&\leq \left(1 - \frac{\underline{\lambda}(Q)}{2\,\overline{\lambda}(P)}\right)(x_k^*)^\top P x_k^* + \frac{(N-1)^2}{N^2}\chi E_k^2,
\end{aligned}
$$

where χ is defined in (22).

Next, from the propagation function (19) and Young's inequality,

$$
\begin{aligned}
E_{k+1}^2 &= \left(\frac{\Lambda}{N} E_k + \sqrt{\phi V_k}\right)^2 \\
&\leq (1+\psi)\frac{\Lambda^2}{N^2} E_k^2 + \left(1 + \frac{1}{\psi}\right)\phi V_k
\end{aligned}
$$

where $\psi > 0$ is chosen according to (23).

Finally, combining both bounds in $V_{k+1} = (x_{k+1}^*)^\top P x_{k+1}^* + \rho E_{k+1}^2$ yields

$$
V_{k+1} \leq \nu V_k,
$$

where ν is defined in (25).

If $c_k = 0$, then $x_{k+1}^* = S c_k = 0$, and

$$
V_{k+1} = \rho E_{k+1}^2 \leq \left((1+\psi)\frac{\Lambda^2}{N^2} + \left(1 + \frac{1}{\psi}\right)\phi\rho\right) V_k,
$$

so (31) still holds.

A.2 Proof of Lemma 2

We first prove (32). Since $V_k = (x_k^*)^\top P x_k^* + \rho E_k^2$, the upper bound in (49) yields

$$\sqrt{V_k} \leq \sqrt{n_x\,\overline{\lambda}(P)}|x_k^*| + \sqrt{\rho}E_k,$$

where we used $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Moreover, by (5),

$$|x_k^*| \leq |x(t_k)| + |x(t_k) - x_k^*| \leq |x(t_k)| + E_k.$$

Hence (32) holds with

$$C_1 := \sqrt{n_x\,\overline{\lambda}(P)} + \sqrt{\rho}.$$

Next, we establish (33). From (5),

$$|x(t_k)| \leq |x_k^*| + |x(t_k) - x_k^*| \leq |x_k^*| + E_k.$$

Using the lower bound in (49), we obtain

$$|x_k^*| \leq \sqrt{\frac{V_k}{\underline{\lambda}(P)}}, \qquad E_k \leq \sqrt{\frac{V_k}{\rho}}$$

which yields (33) with

$$C_2 := \frac{1}{\sqrt{\underline{\lambda}(P)}} + \frac{1}{\sqrt{\rho}}.$$

Finally, we prove (34). From the error estimate (16) and propagation function (18),

$$\begin{aligned} |x(t_{k+1})| &\leq |\hat{x}(t_{k+1}^-)| + |e(t_{k+1}^-)| \\ &\leq \|S\||c_k| + \frac{\Lambda}{N}E_k + \Phi\|d\|_{[t_k,t_{k+1}]}. \end{aligned}$$

We first consider the case $c_k \neq 0$. By (8) and the lower bound in (49),

$$|c_k| \leq |x_k^*| + |c_k - x_k^*| \leq |x_k^*| + \frac{N-1}{N}E_k.$$

Bounding $|x_k^*|$ and E_k in terms of V_k yields (34) with

$$C_3 := \max\left\{ \frac{\|S\|}{\sqrt{\underline{\lambda}(P)}}, \frac{(N-1)\|S\| + \Lambda}{N\sqrt{\rho}} \right\}.$$

If $c_k = 0$, then $\hat{x}(t_{k+1}^-) = Sc_k = 0$, and

$$|x(t_{k+1})| = |e(t_{k+1}^-)| \leq \frac{\Lambda}{N}E_k + \Phi\|d\|_{[t_k,t_{k+1}]},$$

so (34) still holds.

A.3 Proof of Lemma 3

By iterating Lemma 1, we obtain

$$V_{k-1} \le \nu^{k-1-l} V_l,$$

where $\nu \in (0,1)$. Applying (32) at t_l and (34) at t_{k-1} yields

$$\begin{aligned}|x(t_k)| &\le C_3\sqrt{V_{k-1}} + \Phi\|d\|_{[t_{k-1},t_k]}\\ &\le C_3\nu^{(k-l-1)/2}\sqrt{V_l} + \Phi\|d\|_{[t_{k-1},t_k]}\\ &\le C_1C_3\nu^{(k-l-1)/2}(|x(t_l)| + E_l) + \Phi\|d\|_{[t_{k-1},t_k]}.\end{aligned}$$

Hence (35) holds with

$$C := C_1C_3\nu^{-1/2}, \qquad \lambda := -\frac{1}{2}\ln\nu > 0.$$

A.4 Proof of Lemma 4

We consider two cases depending on whether the initial state dominates the disturbance.

Case 1: $|x(t_l)| > \Phi\|d\|_{[t_l,t_k]}$. Let

$$H := 2\|S\| + \Lambda > 1.$$

Then $\log_\nu H < 0$ since $\nu \in (0,1)$. Fix any $\kappa \in (0, -1/\log_\nu H)$, and define

$$l_x := \max\{\lceil \kappa \log_\nu |x(t_l)|\rceil, 0\}.$$

Then $\nu^{l_x} \le |x(t_l)|^\kappa$ for all $|x(t_l)| > 0$. We distinguish between the "long-time" regime $k \ge l + l_x$ and the "short-time" regime $k < l + l_x$.

If $k \ge l + l_x$, iterating Lemma 1 yields

$$V_k \le \nu^{k-l}V_l \le \nu^{l_x}V_l \le |x(t_l)|^\kappa V_l.$$

Applying (32) at t_l and (33) at t_k, we obtain

$$\begin{aligned}|x(t_k)| &\le C_2\sqrt{V_k}\\ &\le C_2|x(t_l)|^{\kappa/2}\sqrt{V_l}\\ &\le C_1C_2|x(t_l)|^{\kappa/2}(|x(t_l)| + E_l) =: \chi_1^x(E_l, |x(t_l)|),\end{aligned}$$

where $\chi_1^x(\cdot, s)$ is nondecreasing for each fixed $s > 0$ and $\chi_1^x(E, \cdot) \in \mathcal{K}_\infty$ for each fixed $E > 0$.

If $l \le k < l + l_x$, we first consider the case $c_k \ne 0$. From the error estimate (16) and propagation function (18),

$$\begin{aligned}|x(t_{k+1})| &\le |\hat{x}(t_{k+1}^-)| + |e(t_{k+1}^-)|\\ &\le \|S\||c_k| + \frac{\Lambda}{N}E_k + \Phi\|d\|_{[t_k,t_{k+1}]}.\end{aligned}$$

By (7),

$$|c_k| \le |x(t_k)| + |x(t_k) - c_k| \le |x(t_k)| + \frac{E_k}{N},$$

and since (6) fails at t_k,

$$|x(t_k)| > \frac{E_k}{N}.$$

Hence

$$|x(t_{k+1})| \le H|x(t_k)| + \Phi\|d\|_{[t_k,t_{k+1}]}. \tag{50}$$

If $c_k = 0$, then $u \equiv 0$ on $[t_k, t_{k+1})$, and

$$\begin{aligned}|x(t_{k+1})| &\le \|e^{A\tau_s}\||x(t_k)| + \left(\int_0^{\tau_s} \|e^{As}D\| \,\mathrm{d}s\right)\|d\|_{[t_k,t_{k+1}]} \\ &\le \Lambda|x(t_k)| + \Phi\|d\|_{[t_k,t_{k+1}]},\end{aligned}$$

so (50) still holds. Since (50) holds for all sampling times between t_l and t_k, iterating it yields

$$\begin{aligned}|x(t_k)| &\le H^{k-l}|x(t_l)| + \frac{H^{k-l}-1}{H-1}\Phi\|d\|_{[t_l,t_k]} \\ &\le \frac{H^{k-l+1}-1}{H-1}|x(t_l)| \\ &\le \frac{H^{l_x}-1}{H-1}|x(t_l)| =: \chi_2^x(|x(t_l)|).\end{aligned}$$

If $|x(t_l)| = s < 1$, then $l_x < \kappa \log_\nu s + 1$, and

$$\chi_2^x(s) < \frac{H^{\kappa \log_\nu s + 1}}{H-1}s = \frac{H}{H-1}s^{\kappa \log_\nu H + 1} =: \chi_3^x(s).$$

where $\chi_3^x \in \mathcal{K}_\infty$ since $\kappa \log_\nu H + 1 > 0$. If $|x(t_l)| = s \ge 1$, then $l_x = 0$, and $\chi_2^x(s) = 0 < \chi_3^x(s)$.

Finally,

$$\begin{aligned}|x(t_k)| &\le \max\{\chi_1^x(E_l, |x(t_l)|),\, \chi_3^x(|x(t_l)|)\} \\ &=: \chi^x(E_l, |x(t_l)|),\end{aligned}$$

which satisfies the stated monotonicity and $\mathcal{K}_\infty$ properties.

Case 2: $|x(t_l)| \le \Phi\|d\|_{[t_l,t_k]}$. The proof follows the same steps as in Case 1 and is omitted for brevity. It yields analogous functions χ_1^d and χ_3^d such that

$$\begin{aligned}|x(t_k)| &\le \max\{\chi_1^d(E_l, \|d\|_{[t_l,t_k]}),\, \chi_3^d(\|d\|_{[t_l,t_k]}|)\} \\ &=: \chi^d(E_l, \|d\|_{[t_l,t_k]}),\end{aligned}$$

which satisfies the stated monotonicity and $\mathcal{K}_\infty$ properties.

Combining the two cases yields (36).

A.5 Proof of Lemma 5

Since (5) holds at t_{j-1} but fails at t_j,

$$|e(t_j)| = |x(t_j) - x_j^*| > E_j.$$

Applying the error estimate (16) and propagation function (19) at t_{j-1}, we obtain

$$\frac{\Lambda}{N} E_{j-1} + \Phi \|d\|_{[t_{j-1}, t_j]} > \frac{\Lambda}{N} E_{j-1} + \sqrt{\phi V_{j-1}},$$

that is,

$$\sqrt{\phi V_{j-1}} < \Phi \|d\|_{[t_{j-1}, t_j]}.$$

Therefore,

$$E_{j-1} \le \sqrt{\frac{V_{j-1}}{\rho}} < \frac{1}{\sqrt{\rho\phi}} \Phi \|d\|_{[t_{j-1}, t_j]}.$$

Next, applying (34) at t_{j-1} yields

$$\begin{aligned} |x(t_j)| &\le C_3 \sqrt{V_{j-1}} + \Phi \|d\|_{[t_{j-1}, t_j]} \\ &< \left(\frac{C_3}{\sqrt{\phi}} + 1 \right) \Phi \|d\|_{[t_{j-1}, t_j]}. \end{aligned}$$

Combining the two bounds, we obtain (37) with

$$\Gamma := \max\left\{ \frac{1}{\sqrt{\rho\phi}}, \frac{C_3}{\sqrt{\phi}} + 1 \right\} \Phi > 0,$$

where $C_3 > 0$ is from Lemma 2.

A.6 Proof of Lemma 6

Let $t_k > 0$ be a sampling time such that the state has not yet been captured by t_k. Then the system remains in a searching stage on $[0, t_k)$, and iterating the error estimate (27) and propagation functions (28) and (29) from 0 to t_k with $x_0^* = 0$ yields

$$\begin{aligned} |e(t_k^-)| &\le \Lambda^k |x_0| + \frac{\Lambda^k - 1}{\Lambda - 1} \Phi \|d\|_{[0, t_k]}, \\ E_k &= \hat{\Lambda}^k E_0 + \frac{\hat{\Lambda}^k - 1}{\hat{\Lambda} - 1} \Phi \delta. \end{aligned}$$

Let i_0^* be the smallest positive integer such that

$$i_0^* \ge \max\left\{ \eta_x\left(\frac{|x_0|}{E_0} \right), \eta_d\left(\frac{\|d\|_{[0, t_{i_0^*}]}}{\delta} \right) \right\}.$$

Such an integer exists if $\|d\|$ is finite. We show that the state is captured no later than $t_{i_0^*}$.

First, since $i_0^* \geq \eta_x(|x_0|/E_0)$,

$$\begin{aligned}\hat{\Lambda}^{i_0^*} E_0 &= \Lambda^{i_0^*}(1+\varepsilon)^{i_0^*} E_0 \\ &\geq \Lambda^{i_0^*}(1+\varepsilon)^{\eta_x(|x_0|/E_0)} E_0 \geq \Lambda^{i_0^*}|x_0|.\end{aligned}$$

Second, if $\delta < \|d\|_{[0,t_{i_0^*}]}$, then

$$i_0^* \geq \eta_d\left(\frac{\|d\|_{[0,t_{i_0^*}]}}{\delta}\right) \geq \log_{1+\varepsilon}\left(\frac{r_\varepsilon \|d\|_{[0,t_{i_0^*}]}}{\delta}\right),$$

and

$$\begin{aligned}\frac{\hat{\Lambda}^{i_0^*}-1}{\hat{\Lambda}-1}\delta &> \frac{\Lambda^{i_0^*}-1}{\hat{\Lambda}-1}(1+\varepsilon)^{i_0^*}\delta \\ &= \frac{\Lambda^{i_0^*}-1}{\Lambda-1}\frac{(1+\varepsilon)^{i_0^*}}{r_\varepsilon}\delta \geq \frac{\Lambda^{i_0^*}-1}{\Lambda-1}\|d\|_{[0,t_{i_0^*}]}.\end{aligned}$$

Otherwise, $\delta \geq \|d\|_{[0,t_{i_0^*}]}$, and since $\hat{\Lambda} > \Lambda$,

$$\frac{\hat{\Lambda}^{i_0^*}-1}{\hat{\Lambda}-1}\delta \geq \frac{\Lambda^{i_0^*}-1}{\Lambda-1}\|d\|_{[0,t_{i_0^*}]}.$$

Combining these bounds yields

$$E_{i_0^*} \geq |e(t_{i_0^*})|.$$

Hence the state is captured at some sampling time $t_{i_0} \leq t_{i_0^*}$. If $i_0 < i_0^*$, then (38) follows immediately. If $i_0 = i_0^*$, then

$$i_0 - 1 < \max\left\{\eta_x\left(\frac{|x_0|}{E_0}\right), \eta_d\left(\frac{\|d\|_{[0,t_{i_0-1}]}}{\delta}\right)\right\},$$

and since both sides are integers, (38) still holds.

A.7 Proof of Lemma 7

Since Consider any $t_k \leq t_{i_0}$. the state is first captured at t_{i_0}, the system remains in a searching stage on $[0, t_{i_0})$.

First, iterating the error estimate (27) and propagation function (28) from 0 to t_k with $x_0^* = 0$ yields

$$\begin{aligned}|x(t_k)| &\leq \Lambda^k|x_0| + \frac{\Lambda^k - 1}{\Lambda - 1}\Phi\|d\|_{[0,t_k]} \\ &\leq \Lambda^{i_0}|x_0| + \frac{\Lambda^{i_0}-1}{\Lambda-1}\Phi\|d\|_{[0,t_{i_0}]}.\end{aligned}$$

Substituting the bound on i_0 from (38) into the right-hand side, we obtain a bound on $|x(t_k)|$ that is nondecreasing in $|x_0|$ and $\|d\|_{[0,t_{i_0}]}$ and vanishes when $|x_0| = \|d\|_{[0,t_{i_0}]} = 0$. Therefore, it can be dominated by $\mathcal{K}_\infty$ functions $\hat{\gamma}_0^x(|x_0|) + \hat{\gamma}_0^d(\|d\|_{[0,t_{i_0}]})$ using a standard comparison-function construction, which proves (39).

Second, iterating the propagation function (29) from 0 to t_{i_0} yields

$$E_{i_0} = \hat{\Lambda}^{i_0} E_0 + \frac{\hat{\Lambda}^{i_0} - 1}{\hat{\Lambda} - 1} \Phi\delta.$$

Substituting again the bound on i_0 from (38) into the right-hand side, we obtain a bound on E_{i_0} that, for each fixed $E_0 > 0$, is nondecreasing in $|x_0|$ and $\|d\|_{[0,t_{i_0}]}$. Therefore, by another standard comparison-function construction, there exists a continuous function $\hat{\chi}_0^E : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ with the stated monotonicity property such that (40) holds.

A.8 Proof of Lemma 8

If the state is recaptured at t_{j+1}, then $i = j + 1$ and (41) holds. In the remainder of the proof, we assume that the state is not recaptured at t_{j+1}.

The system remains in a searching stage on $[t_j, t_i)$. Applying the same argument as in the proof of Lemma 6, with the adjustment at t_j according to (30), we obtain

$$i \leq j + \max\left\{ \eta_x\left(\frac{|e(t_j)|}{\hat{E}_j}\right),\, \eta_d\left(\frac{\|d\|_{[t_j,t_i]}}{\delta}\right),\, 1 \right\}, \tag{51}$$

where η_x, η_d are defined in Lemma 6 and satisfy $\eta_x(s) \leq \eta_d(s)$ for all $s \geq 0$.

Next, applying the searching-stage error estimate (27) at t_j and the stabilizing-stage error estimate (16) at t_{j-1} yields

$$\begin{aligned} |e(t_{j+1})| &\leq \Lambda |e(t_j)| + \Phi \|d\|_{[t_j,t_{j+1}]} \\ &\leq \frac{\Lambda^2}{N} |e(t_{j-1})| + (\Lambda + 1)\Phi \|d\|_{[t_{j-1},t_{j+1}]}. \end{aligned}$$

Moreover, from the adjusted propagation function (30),

$$E_{j+1} = \hat{\Lambda}\hat{E}_j + \Phi\delta = \frac{\hat{\Lambda}\Lambda}{N} E_{j-1} + (\hat{\Lambda} + 1)\Phi\delta,$$

where $\hat{\Lambda} = (1 + \varepsilon)\Lambda$.

Since the state escapes at t_j and is not recaptured at t_{j+1},

$$|e(t_{j-1})| \leq E_{j-1}, \qquad |e(t_{j+1})| > E_{j+1},$$

and the above bounds imply

$$\delta < \|d\|_{[t_{j-1},t_{j+1}]}.$$

Consequently,

$$\frac{|e(t_j)|}{\hat{E}_j} \leq \frac{\frac{\Lambda}{N}|e(t_{j-1})| + \Phi\|d\|_{[t_{j-1},t_{j+1}]}}{\frac{\Lambda}{N} E_{j-1} + \Phi\delta} < \frac{\|d\|_{[t_{j-1},t_{j+1}]}}{\delta},$$

and thus

$$\eta_x\left(\frac{|e(t_j)|}{\hat{E}_j}\right) \leq \eta_x\left(\frac{\|d\|_{[t_{j-1},t_{j+1}]}}{\delta}\right) \leq \eta_d\left(\frac{\|d\|_{[t_{j-1},t_{j+1}]}}{\delta}\right).$$

Substituting this bound into (51) yields (41).

A.9 Proof of Lemma 9

Since the state escapes at t_j and is recaptured at t_i, the system remains in a searching stage on $[t_j, t_i)$. Consider any sampling time $t_k \in [t_j, t_i)$.

First, since $u \equiv 0$ on $[t_j, t_i)$, the state satisfies

$$\dot{x} = Ax + Dd,$$

and thus

$$|x(t_{k+1})| \leq \Lambda |x(t_k)| + \Phi \|d\|_{[t_k, t_{k+1}]}.$$

Iterating this bound from t_j to t_k yields

$$\begin{aligned} |x(t_k)| &\leq \Lambda^{k-j} |x(t_j)| + \frac{\Lambda^{k-j} - 1}{\Lambda - 1} \Phi \|d\|_{[t_j, t_k]} \\ &\leq \Lambda^{i-j} |x(t_j)| + \frac{\Lambda^{i-j} - 1}{\Lambda - 1} \Phi \|d\|_{[t_j, t_i]}. \end{aligned}$$

Substituting the bound on i from (41):

$$i \leq j + \max\left\{ \eta_d \left(\frac{\|d\|_{[t_{j-1}, t_i]}}{\delta} \right), 1 \right\},$$

into the right-hand side, we obtain a bound on $|x(t_k)|$ that is nondecreasing in $|x(t_j)|$ and $\|d\|_{[t_{j-1}, t_i]}$ and vanishes when $|x(t_j)| = \|d\|_{[t_{j-1}, t_i]} = 0$. Therefore, it can be dominated by $\mathcal{K}_\infty$ functions $\hat{\gamma}^x(|x(t_j)|) + \hat{\gamma}^d(\|d\|_{[t_{j-1}, t_i]})$ using a standard comparison-function construction, which proves (42).

Second, iterating the propagation function (29) from t_j to t_i, with the adjustment at t_j according to (30), yields

$$\begin{aligned} E_i &= \hat{\Lambda}^{i-j} \hat{E}_j + \frac{\hat{\Lambda}^{i-j} - 1}{\hat{\Lambda} - 1} \Phi \delta \\ &= \frac{\hat{\Lambda}^{i-j} \Lambda}{N} E_{j-1} + \frac{\hat{\Lambda}^{i-j+1} - 1}{\hat{\Lambda} - 1} \Phi \delta. \end{aligned}$$

Substituting again the bound on i from (41), we obtain a bound on E_i that, for each fixed $E_{j-1} > 0$, is nondecreasing in $\|d\|_{[t_{j-1}, t_i]}$. Therefore, by another standard comparison-function construction, there exists a continuous function $\hat{\chi}^E : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ with the stated monotonicity property such that (43) holds.

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