Analysis of different Lyapunov function constructions for interconnected hybrid systems

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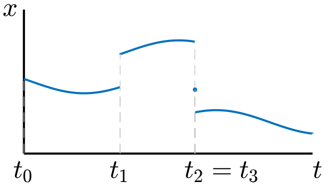
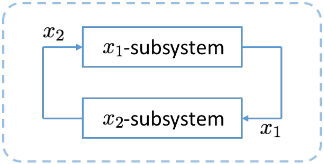
Introduction

- Hybrid systems: dynamical systems exhibiting both continuous and discrete behaviors



- Modeling framework [GST12; CT09]

- Interconnected hybrid systems



- Generalized ISS Lyapunov function for each subsystem
- Small-gain conditions (SG)
- Non-ISS dynamics in subsystems

[GST12] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012

[CT09] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. & Control Lett.*, vol. 58, no. 1, pp. 47–53, 2009

Literature review

- Non-ISS jumps: *average dwell-time (ADT)* [HM99]
- Non-ISS flows: *reverse ADT (RADT)* [HLT08]
- Strategy 1 [LNT14]: increase feedback gains
  1. ADT/RADT modifications for ISS Lyapunov functions for subsystems
  2. SG for stability of the interconnection
- Strategy 2 [Das+12]: cannot apply to mixed non-ISS dynamics
  1. SG for a generalized Lyapunov function for the interconnection
  2. ADT/RADT modification for stability of the interconnection
- In this work, we provide a thorough study on
  - the effects of ADT/RADT modifications on feedback gains
  - the applicability of the two strategies

[HM99] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *38th IEEE Conf. Decis. Control*, vol. 3, 1999, pp. 2655–2660

[HLT08] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, 2008

[LNT14] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based small-gain theorems for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 59, no. 6, pp. 1395–1410, 2014

[Das+12] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, "Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods," *Nonlinear Anal. Hybrid Syst.*, vol. 6, no. 3, pp. 899–915, 2012

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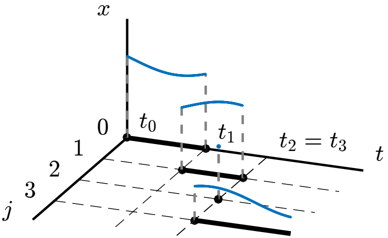
Hybrid system with input

$$\dot{x} \in F(x,u), \qquad (x,u) \in \mathcal{C},$$
$$x^+ \in G(x,u), \qquad (x,u) \in \mathcal{D}.$$

- State $x \in \mathcal{X} \subset \mathbb{R}^n$, input $u \in \mathcal{U} \subset \mathbb{R}^m$
- *Flow set* $\mathcal{C} \subset \mathcal{X} \times \mathcal{U}$, *flow map* $F : \mathcal{X} \times \mathcal{U} \rightrightarrows \mathbb{R}^n$
- *Jump set* $\mathcal{D} \subset \mathcal{X} \times \mathcal{U}$, *jump map* $G : \mathcal{X} \times \mathcal{U} \rightrightarrows \mathcal{X}$
- Solutions $x : \operatorname{dom} x \to \mathcal{X}$ defined on hybrid time domains

$$\operatorname{dom} x = \bigcup_{j=0,1,\dots} [t_j, t_{j+1}] \times \{j\}$$



[CT09] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. & Control Lett.*, vol. 58, no. 1, pp. 47–53, 2009

Input-to-state stability

$$\dot{x} \in F(x,u), \qquad (x,u) \in \mathcal{C},$$
$$x^+ \in G(x,u), \qquad (x,u) \in \mathcal{D}.$$

- State $x \in \mathcal{X} \subset \mathbb{R}^n$, input $u \in \mathcal{U} \subset \mathbb{R}^m$

Definition

A hybrid system is input-to-state stable (ISS) w.r.t. a set $\mathcal{A} \subset \mathcal{X}$ if there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that all solution pairs (x,u) satisfy

$$|x(t,j)|_\mathcal{A} \leq \beta(|x(0,0)|_\mathcal{A}, t+j) + \gamma(\|u\|_{(t,j)}) \qquad \forall\,(t,j) \in \operatorname{dom} x.$$

- In the absence of inputs, ISS becomes global asymptotic stability (GAS)

[Son89] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, no. 4, pp. 435–443, 1989

Candidate ISS Lyapunov function

Definition 1

A locally Lipschitz function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a candidate ISS Lyapunov function w.r.t $\mathcal{A}$ if

1. $\exists$ bounds $\psi_1, \psi_2 \in \mathcal{K}_\infty$ s.t. $\psi_1(|x|_\mathcal{A}) \leq V(x) \leq \psi_2(|x|_\mathcal{A})$ for all $x \in \mathcal{X}$;
2. $\exists$ an input gain $\chi \in \mathcal{K}_\infty$ and a rate $\phi \in C^0(\mathbb{R}_{\geq 0}, \mathbb{R})$ with $\phi(0) = 0$ s.t.
$$V(x) \geq \chi(|u|) \Rightarrow \nabla_v V(x) \leq -\phi(V(x)) \qquad \forall\,(x,u) \in \mathcal{C}, \forall\, v \in F(x,u);$$
3. $\exists$ a positive definite rate $\alpha \in C^0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ such that
$$V(y) \leq \max\{\alpha(V(x)), \chi(|u|)\} \qquad \forall\,(x,u) \in \mathcal{D}, \forall\, y \in G(x,u).$$

It is an ISS Lyapunov function if $\phi(r) > 0$ and $\alpha(r) < r$ for all $r > 0$.

Proposition 1 ([CT09, Prop. 2.7])

A hybrid system is ISS if it admits an ISS Lyapunov function.

[CT09] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. & Control Lett.*, vol. 58, no. 1, pp. 47–53, 2009

[CT13] C. Cai and A. R. Teel, "Robust input-to-state stability for hybrid systems," *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 1651–1678, 2013

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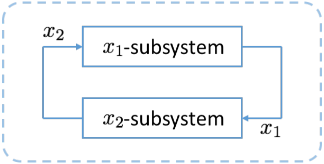
Interconnection of two hybrid subsystems

- Hybrid system with state $x = (x_1, x_2)$

$$\dot{x}_1 = f_1(x), \dot{x}_2 = f_2(x), \quad x \in \mathcal{C},$$
$$x_1^+ = g_1(x), x_2^+ = g_2(x), \quad x \in \mathcal{D}.$$

- Each x_i-subsystem regards x_j as an input



Assumption 1

Each x_i-subsystem (with input x_j) admits a candidate ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ with bounds ψ_{1i}, ψ_{2i}, an input gain χ_i, and rates ϕ_i, α_i.

- For all $x = (x_1, x_2) \in \mathcal{C}$,

$$V_1(x_1) \geq \gamma_1(V_2(x_2)) \quad \Rightarrow \quad \nabla_{f_1(x)} V_1(x_1) \leq -\phi_1(V_1(x_1)),$$
$$V_2(x_2) \geq \gamma_2(V_1(x_1)) \quad \Rightarrow \quad \nabla_{f_2(x)} V_2(x_2) \leq -\phi_2(V_2(x_2))$$

with $\gamma_i(r) := \chi_i(\psi_{ij}^{-1}(r))$ for $i = 1, 2$

- Small-gain condition (SG): the composition $\gamma_1 \circ \gamma_2 < \mathrm{Id}$

Small-gain theorem

Assumption 1

Each x_i-subsystem (with input x_j) admits a candidate ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ with bounds ψ_{1i}, ψ_{2i}, a gain χ_i, and rates ϕ_i, α_i.

(SG1) The composition $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := \chi_i(\psi_{1j}^{-1}(r))$ for $i = 1, 2$.

Proposition 2

Suppose Assumption 1 and (SG1) hold. Then $V(x) := \max\{\rho(V_1(x_1)), V_2(x_2)\}$ with ρ in Lemma 1 is a candidate Lyapunov function for the interconnection.

Proposition 3 ([LNT14, Th. III.1 and Cor. III.2])

Suppose Assumption 1 and (SG1) hold with ISS Lyapunov functions V_1, V_2. Then V defined in Proposition 2 is a Lyapunov function and ensures GAS.

[LNT14] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based small-gain theorems for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 59, no. 6, pp. 1395–1410, 2014

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Candidate exponential ISS Lyapunov functions

Definition 2

A candidate ISS Lyapunov function with rates ϕ, α satisfying

$$\phi(r) \equiv cr, \qquad \alpha(r) \equiv e^{-d} r$$

for some constants $c, d \in \mathbb{R}$ is a candidate exponential ISS Lyapunov function with rate coefficients c, d.
It is an exponential ISS Lyapunov function if $c, d > 0$.

Assumption 2

Each x_i-subsystem admits a candidate exponential ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ with bounds ψ_{1i}, ψ_{2i}, a gain χ_i, and rate coefficients c_i, d_i.

Non-ISS flows: reverse average dwell-time (RADT)

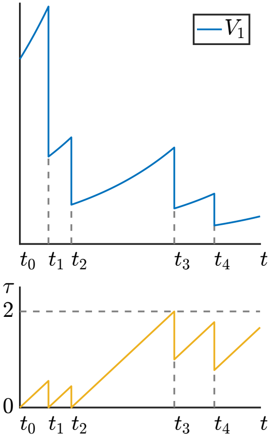
- Assumption 2 holds with V_1, V_2 and rate coefficients $c_1, c_2 \leq 0 < d_1, d_2$
- Consider solutions that jump fast enough
- A solution x admits a reverse average dwell-time (RADT) $\tau_a^* > 0$ [HLT08] if

$$j - k \geq (t-s)/\tau_a^* - N_0^* \qquad \forall\, t \geq s$$

with an integer $N_0^* \geq 1$.

- [CTG08] Equivalently, $\operatorname{dom} x = \operatorname{dom} \tau$ for an RADT timer τ with

$$\dot{\tau} = 1/\tau_a^*, \qquad \tau \in [0, N_0^*],$$
$$\tau^+ = \max\{0, \tau - 1\}, \quad \tau \in [0, N_0^*].$$



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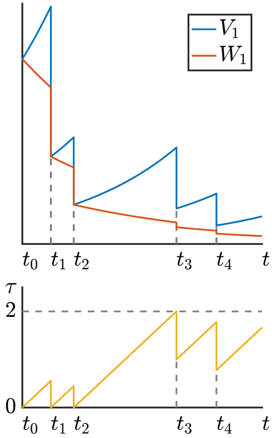
[CTG08] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems Part II: (Pre)Asymptotically stable compact sets," IEEE Trans. Automat. Contr., vol. 53, no. 3, pp. 734–748, 2008

Non-ISS flows: RADT modification

- Assumption 2 holds with V_1, V_2 and rate coefficients $c_1, c_2 \leq 0 < d_1, d_2$
- Consider solutions that jump fast enough
- Consider the augmented interconnection with state (x_1, x_2, τ)
- [MYL14, Prop. 6] Provided that $\tau_a^* < -d_i/c_i$, there exists an $L_i \in (-c_i\tau_a^*, d_i)$ s.t.

$$W_i(x_i, \tau) := e^{-L_i \tau} V_i(x_i)$$

  is an exponential ISS Lyapunov function
- To establish GAS via SG, it requires $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := \chi_i(\psi_{1j}^{-1}(e^{L_j N_0^*} r))$ for $i = 1, 2$



[MYL14] A. Mironchenko, G. Yang, and D. Liberzon, "Lyapunov small-gain theorems for not necessarily ISS hybrid systems," in *21st Int. Symp. Math. Theory Networks Syst.*, 2014, pp. 1001–1008

Non-ISS flows: small-gain theorem

- Equivalently:

(SG2) There exists an $\varepsilon > 0$ such that $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := \chi_i(\psi_{1j}^{-1}((1+\varepsilon)r))$ for $i = 1, 2$.

Theorem 5

Suppose Assumption 2 holds with rate coefficients $c_1, c_2 \leq 0 < d_1, d_2$. Provided that (SG2) is satisfied, the GAS estimate holds for every solution with a small enough RADT.

- Before RADT modification

(SG1) The composition $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := \chi_i(\psi_{1j}^{-1}(r))$ for $i = 1, 2$.

- (SG2) is generic in (SG1) (in particular, they are equivalent for linear gains)
- RADT modification does not substantially increase the feedback gains

Non-ISS jumps: average dwell-time (ADT)

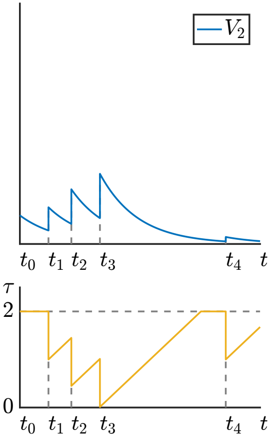
- Assumption 2 holds with V_1, V_2 and rate coefficients $c_1, c_2 > 0 \geq d_1, d_2$
- Consider solutions that jump slowly enough
- A solution x admits an average dwell-time (ADT) $\tau_a > 0$ [HM99] if

$$j - k \leq (t - s)/\tau_a + N_0 \qquad \forall\, t \geq s$$

with an integer $N_0^* \geq 1$.

- [CTG08] Equivalently, $\operatorname{dom} x = \operatorname{dom} \tau$ for an ADT timer τ with

$$\dot{\tau} = [0, 1/\tau_a], \quad \tau \in [0, N_0],$$
$$\tau^+ = \tau - 1, \quad \tau \in [1, N_0].$$



[HM99] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *38th IEEE Conf. Decis. Control*, vol. 3, 1999, pp. 2655–2660

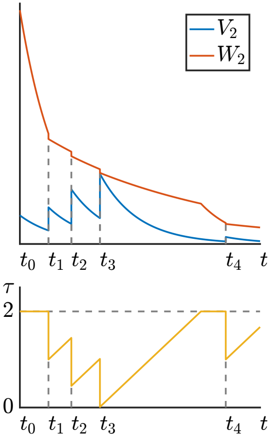
[CTG08] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems Part II: (Pre)Asymptotically stable compact sets," *IEEE Trans. Automat. Contr.*, vol. 53, no. 3, pp. 734–748, 2008

Non-ISS jumps: ADT modification

- Assumption 2 holds with V_1, V_2 and rate coefficients $c_1, c_2 > 0 \geq d_1, d_2$
- Consider solutions that jump slowly enough
- Consider the augmented interconnection with state (x_1, x_2, τ)
- [MYL14, Prop. 5] Provided that $\tau_a > -d_i/c_i$, there exists an $L_i \in (-d_i, c_i\tau_a)$ s.t.

$$W_i(x_i, \tau) := e^{L_i \tau} V_i(x_i)$$

  is an exponential ISS Lyapunov function
- To establish GAS via SG, it requires $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := e^{L_i N_0} \chi_i(\psi_{1j}^{-1}(r))$ for $i = 1, 2$



[MYL14] A. Mironchenko, G. Yang, and D. Liberzon, "Lyapunov small-gain theorems for not necessarily ISS hybrid systems," in *21st Int. Symp. Math. Theory Networks Syst.*, 2014, pp. 1001–1008

Non-ISS jumps: small-gain theorem

- Equivalently:

(SG3) There exists an $\varepsilon > 0$ such that $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := (1+\varepsilon)e^{-d_i}\chi_i(\psi_{1j}^{-1}(r))$ for $i = 1, 2$.

Theorem 6

Suppose Assumption 2 holds with rate coefficients $c_1, c_2 > 0 \geq d_1, d_2$. Provided that (SG3) is satisfied, the GAS estimate holds for every solution with a large enough ADT.

- Before ADT modification

(SG1) The composition $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ with $\gamma_i(r) := \chi_i(\psi_{1j}^{-1}(r))$ for $i = 1, 2$.

- Unlike (SG2) for RADT, (SG3) is not generic in (SG1)
- ADT modification substantially increases the feedback gains

Non-ISS jumps: an alternate construction

- Assumption 2 holds with rate coefficients $c_1, c_2 > 0 \geq d_1, d_2$
- Linear gains: $\gamma_1(r) \equiv \xi_1 r$ and $\gamma_2(r) \equiv \xi_2 r$ for some constant $\xi_1, \xi_2 > 0$
- Construct a candidate exponential Lyapunov function for the interconnection
- Establish GAS under ADT
- Advantage: it requires (SG1: $\xi_1 \xi_2 < 1$) instead of (SG3: $\xi_1 \xi_2 < e^{d_1+d_2}$)
- Disadvantage: it requires linear gains

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Summary

- Stability of interconnected hybrid systems
- Lyapunov function constructions based on small-gain conditions
- Non-ISS subsystems: ADT/RADT modifications

Dynamics	Small-gain condition	Remark
ISS subsystems	(SG1)	
Non-ISS flows	(SG2)	Generic in (SG1)
Non-ISS jumps	(SG3)	Not generic in (SG1)
Non-ISS jumps	(SG1)	Linear gains
Non-ISS flow and jump	(SG4)	Not generic in (SG1)
Non-ISS flow and jump	(SG1)	Linear gains

Future research topics

- Generalization for hybrid network of more than 2 subsystems
- Modifying non-exponential ISS Lyapunov functions