Stabilization of interconnected switched control-affine systems via a Lyapunov-based small-gain approach

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Switched system

- Switching is ubiquitous in realistic system models, such as
  - Thermostat
  - Gear transmission
  - Power supply

- Structure of a switched system
  - A family of dynamics, called modes
  - A sequence of events, called switches

- In this work: time-dependent, uncontrolled switching
Presentation outline

- Preliminaries
- Interconnected switched systems and small-gain theorem
- Stabilization via a small-gain approach
Nonlinear switched system with input

\[ \dot{x} = f_\sigma(x, w), \quad x(0) = x_0 \]

- State \( x \in \mathbb{R}^n \), disturbance \( w \in \mathbb{R}^m \)
- A family of modes \( f_p, p \in \mathcal{P} \), with an index set \( \mathcal{P} \)
- A right-continuous, piecewise constant switching signal \( \sigma : \mathbb{R}_+ \to \mathcal{P} \) that indicates the active mode \( \sigma(t) \)
- Solution \( x(\cdot) \) is absolutely continuous (no state jump)
Stability notions

Definition (GAS)

A continuous-time system is globally asymptotically stable (GAS) if there is a function $\beta \in KL$ s.t. for all initial state $x_0$,

$$|x(t)| \leq \beta(|x_0|, t) \quad \forall t \geq 0.$$ 

A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $K$ if it is continuous, positive definite and strictly increasing; $\alpha \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if $\lim_{r \to \infty} \alpha(r) = \infty$, such as $\alpha(r) = r^2$ or $|r|$.

A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{t \to \infty} \gamma(t) = 0$, such as $\gamma(t) = e^{-t}$.

A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $KL$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t$, and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed $r > 0$, such as $\beta(r, t) = r^2 e^{-t}$.
Stability notions

Definition (GAS)

A continuous-time system is globally asymptotically stable (GAS) if there is a function $\beta \in \mathcal{KL}$ s.t. for all initial state $x_0$,

$$|x(t)| \leq \beta(|x_0|, t) \quad \forall t \geq 0.$$  

- The switched system may be unstable even if all individual modes are GAS
Stability notions

Definition (GAS)
A continuous-time system is globally asymptotically stable (GAS) if there is a function $\beta \in \mathcal{KL}$ s.t. for all initial state $x_0$,

$$|x(t)| \leq \beta(|x_0|, t) \quad \forall t \geq 0.$$ 

Definition (ISpS [JTP94])
A continuous-time system is input-to-state practically stable (ISpS) if there are functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and a constant $\varepsilon \geq 0$ s.t. for all $x_0$ and disturbance $w$,

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|w\|) + \varepsilon \quad \forall t \geq 0.$$ 

- When $\varepsilon = 0$, ISpS becomes input-to-state stability (ISS) [Son89]
- When $\varepsilon = 0$ and $\gamma \equiv 0$, ISpS becomes GAS

Lyapunov characterizations

\[ \dot{x} = f_\sigma(x, w), \quad x(0) = x_0 \]

- A common Lyapunov function

The switched system is GAS if
- it admits a Lyapunov function \( V \) which decreases along the solution in all modes:

\[ D_{f_p}V(x, w) \leq -\lambda V(x) \]

with a constant \( \lambda > 0 \).

Lyapunov characterizations

\[ \dot{x} = f_\sigma(x, w), \quad x(0) = x_0 \]

- A common Lyapunov function
- Multiple Lyapunov functions

The switched system is GAS if

- each mode admits a Lyapunov function \( V_p \) which decreases along the solution when that mode is active:
  \[ D f_\sigma V_p(x, w) \leq -\lambda V_p(x), \]
- and their values at switches are decreasing:
  \[ V_\sigma(t_k)(x(t_k)) \leq V_\sigma(t_l)(x(t_l)) \]
  for all switches \( t_k > t_l \).

Lyapunov characterizations

\[ \dot{x} = f_\sigma(x, w), \quad x(0) = x_0 \]

- A common Lyapunov function
- Multiple Lyapunov functions
- Dwell-time [Mor96], Average dwell-time (ADT) [HM99]

The switched system is GAS if
- each mode admits a Lyapunov function \( V_p \) which decreases along the solution when that mode is active:
  \[ D_{f_p} V_p(x, w) \leq -\lambda V_p(x), \]
- their values after each switch is bounded in ratio: \( \exists \mu \geq 1 \) s.t.
  \[ V_p(x) \leq \mu V_q(x), \]
- there is an ADT \( \tau_\alpha > \ln(\mu)/\lambda \) with an integer \( N_0 \geq 1 \):
  \[ N_\sigma(t, \tau) \leq N_0 + (t - \tau)/\tau_\alpha. \]

**Lyapunov characterizations**

\[ \dot{x} = f_\sigma(x, w), \quad x(0) = x_0 \]

- A common Lyapunov function
- Multiple Lyapunov functions
- Dwell-time, ADT
- Further results under slow switching:
  - ISS with dwell-time [XWL01]
  - ISS and integral-ISS with ADT [VCL07]
  - ISS and IOSS with ADT [ML12]

The switched system is GAS if

- each mode admits a Lyapunov function \( V_p \) which decreases along the solution when that mode is active:

\[ D_{f_p} V_p(x, w) \leq -\lambda V_p(x), \]

- their values after each switch is bounded in ratio: \( \exists \mu \geq 1 \) s.t.

\[ V_p(x) \leq \mu V_q(x), \]

- there is an ADT \( \tau_a > \ln(\mu)/\lambda \) with an integer \( N_0 \geq 1 \):

\[ N_\sigma(t, \tau) \leq N_0 + (t - \tau)/\tau_a. \]

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Interconnected switched systems

- An interconnection of switched systems with state $x = (x_1, x_2)$ and external disturbance $w$
  \[
  \dot{x}_1 = f_{1,\sigma_1}(x_1, x_2, w), \\
  \dot{x}_2 = f_{2,\sigma_2}(x_1, x_2, w).
  \]

- Each $x_i$-subsystem regards $x_j$ as internal disturbance

- The switchings $\sigma_1, \sigma_2$ are independent

- Each $x_i$-subsystem has stabilizing modes in $\mathcal{P}_{s,i}$ and destabilizing ones in $\mathcal{P}_{u,i}$

- Objective: establish ISpS of the interconnection using
  - Generalized ISpS-Lyapunov functions
  - Average dwell-times (ADT)
  - Time-ratios
  - A small-gain condition
Assumptions

- (Generalized ISpS-Lyapunov) For each $x_i$-subsystem
  - Each $p_s \in \mathcal{P}_{s,i}$ admits an ISpS-Lya function $V_{i,p_s}$ that decreases when active and
    \[
    V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)), \chi_i^w(|w|), \delta_i\};
    \]
  - Each $p_u \in \mathcal{P}_{u,i}$ admits a function $V_{i,p_u}$ that may increase when active;
  - $V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i)$ for all $p, q \in \mathcal{P}_i$.

- (ADT) There is a large enough ADT $\tau_{a,i}$.

- (Time-ratio) There is a small enough time-ratio $\rho_i \in [0, 1)$.
Lyapunov-based small-gain theorem

- (Generalized ISpS-Lyapunov) For each $x_i$-subsystem
  - Each $p_s \in \mathcal{P}_{s,i}$ admits an ISpS-Lya function $V_{i,p_s}$ that decreases when active and
    \[ V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)), \chi^w(|w|), \delta_i\}; \]
  - Each $p_u \in \mathcal{P}_{u,i}$ admits a function $V_{i,p_u}$ that may increase when active;
  - $V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i)$ for all $p, q \in \mathcal{P}_i$.

- (ADT) There is a large enough ADT $\tau_{a,i}$.

- (Time-ratio) There is a small enough time-ratio $\rho_i \in [0, 1)$.

Theorem (Small-Gain)

The interconnection is ISpS provided that the small-gain condition

\[ \chi_1^* \circ \chi_2^* < \Id \]

holds with $\chi_i^* := e^{\Theta_i} \chi_i$. 

Proof: Hybrid ISpS-Lyapunov function

- (Generalized ISpS-Lyapunov) For each $x_i$-subsystem
  - Each $p_s \in P_{s,i}$ admits an ISpS-Lya function $V_{i,p_s}$ that decreases when active and
    $$V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)), \chi_i^w(|w|), \delta_i\};$$
  - Each $p_u \in P_{u,i}$ admits a function $V_{i,p_u}$ that may increase when active;
  - $V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i)$ for all $p, q \in P_i$.

- (ADT) There is a large enough ADT $\tau_{a,i}$.

- (Time-ratio) There is a small enough time-ratio $\rho_i \in [0, 1)$.

- Auxiliary timer $\tau_i \in [0, \Theta_i]$

- Hybrid ISpS-Lya for subsystem $V_i := V_{i,\sigma_i} \phi_i$ with $\chi_i^*: = e^{\Theta_i} \chi_i$
Proof: Hybrid ISpS-Lyapunov function

- (Generalized ISpS-Lyapunov) For each $x_i$-subsystem
  - Each $p_s \in \mathcal{P}_{s,i}$ admits an ISpS-Lya function $V_{i,p_s}$ that decreases when active and
    $$V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)), \chi_i^w(|w|), \delta_i\};$$
  - Each $p_u \in \mathcal{P}_{u,i}$ admits a function $V_{i,p_u}$ that may increase when active;
  - $V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i)$ for all $p, q \in \mathcal{P}_i$.

- (ADT) There is a large enough ADT $\tau_{a,i}$.

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- Auxiliary timer $\tau_i \in [0, \Theta_i]$

- Hybrid ISpS-Lya for subsystem $V_i := V_{i,\sigma_i} e^{\tau_i}$
  with $\chi_i^* := e^{\Theta_i} \chi_i$

- Small-gain $\chi_1^* \circ \chi_2^* < \text{Id}$

- Hybrid ISpS-Lya function
  $$V := \max\{\psi(V_1), V_2\}$$

Proof: Hybrid ISpS-Lyapunov function

- (Generalized ISpS-Lyapunov) For each $x_i$-subsystem
  - Each $p_s \in P_{s,i}$ admits an ISpS-Lya function $V_{i,p_s}$ that decreases when active and
    \[ V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)), \chi_i^w(|w|), \delta_i\}; \]
  - Each $p_u \in P_{u,i}$ admits a function $V_{i,p_u}$ that may increase when active;
  - $V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i)$ for all $p, q \in P_i$.

- (ADT) There is a large enough ADT $\tau_{a,i}$.

- (Time-ratio) There is a small enough time-ratio $\rho_i \in [0, 1)$.

- Auxiliary timer $\tau_i \in [0, \Theta_i]$

- Hybrid ISpS-Lya for subsystem $V_i := V_{i,\sigma_i} e^{\tau_i}$ with $\chi_i^* := e^{\Theta_i} \chi_i$

- Small-gain $\chi_1^* \circ \chi_2^* < \text{Id}$

- Hybrid ISpS-Lya function $V := \max\{\psi(V_1), V_2\}$
Stabilization via a small-gain approach

- An interconnection of switched control-affine systems with state
  \( x = (x_1, x_2) \), external disturbance \( w \) and controls \( u_1, u_2 \)
  
  \[
  \dot{x}_1 = f_{1, \sigma_1}^0(x, w) + G_{1, \sigma_1}(x, w) u_1, \\
  \dot{x}_2 = f_{2, \sigma_2}^0(x, w) + G_{2, \sigma_2}(x, w) u_2. 
  \]

- (Generalized ISS-Lyapunov) For each \( x_i \)-subsystem in open-loop
  
  - Each \( \mathcal{P}_{s,i} \) admits an ISS-Lyap function \( V_{i,p_s} \) that decreases when active and
    
    \[
    V_{i,p_s}(x_i) \geq \max\{\chi_i^0(V_{j,p_j}(x_j)), \chi_i^w(|w|)\};
    \]

Proposition (Gain assignment)

For each \( \chi_i \in \mathcal{K}_\infty \) and \( \delta_i > 0 \), there is a feedback control \( u_i = -\nu_{i, \sigma_i}(|x_i|) \nabla V_{i, \sigma_i} \) s.t. in closed-loop, each \( V_{i,p_s} \) decreases when active and

\[
V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)), \chi_i^w(|w|), \delta_i\}.
\]

Stabilization via a small-gain approach

- An interconnection of switched control-affine systems with state $x = (x_1, x_2)$, external disturbance $w$ and controls $u_1, u_2$

$$\begin{align*}
\dot{x}_1 &= f_{1,\sigma_1}^0(x, w) + G_{1,\sigma_1}(x, w) u_1, \\
\dot{x}_2 &= f_{2,\sigma_2}^0(x, w) + G_{2,\sigma_2}(x, w) u_2.
\end{align*}$$

- (Generalized ISS-Lyapunov) For each $x_i$-subsystem in open-loop
  - Each $p_s \in P_{s,i}$ admits an ISS-Lya function $V_{i,p_s}$ that decreases when active and
    \[ V_{i,p_s}(x_i) \geq \max\{\chi_0^i(V_{j,p_j}(x_j)), \chi_w^i(|w|)\}; \]

**Theorem**

For each $\varepsilon > 0$, there are feedback controls $u_1, u_2$ s.t. the interconnection in closed-loop is ISpS with the constant $\varepsilon$, i.e., there are $\beta \in KL$, $\gamma \in K_\infty$ s.t.

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|w\|) + \varepsilon \quad \forall t \geq 0.$$
Summary

- Interconnected switched systems
- A Lyapunov-based small-gain theorem for ISpS
  - Hybrid ISpS-Lyapunov functions and a small-gain condition
  - Increase in internal gains due to switching and destabilizing modes
- Stabilization via a small-gain approach
  - Gain assignment using feedback control