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Stabilization of interconnected switched control-affine systems via a Lyapunov-based small-gain approach

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Abstract— **We study the feedback stabilization of interconnected switched control-affine systems with both input-to-state stable (ISS) and non-ISS modes. Provided that the switching is slow in the sense of average dwell-time and the active time of non-ISS modes is short in proportion, suitable feedback controls are designed to achieve input-to-state practical stability (ISpS) with an arbitrarily small constant. We devise such feedback controls by extending a previous small-gain theorem on stability of interconnected switched systems to the ISpS context, and proposing a novel Lyapunov-based gain-assignment scheme.**

I. Introduction

In studying real-world phenomena, one usually finds it effective to transform a complex system into an interconnection of simpler subsystems, and establish stability based on properties of the constituents via small-gain theorems. In the input-output context, classical small-gain theorems for linear systems were summarized in [1], and their generalizations for nonlinear feedback interconnections were established in [2], [3]. In recent works on interconnections, the notion of input-to-state stability (ISS) [4] was widely used as it naturally unifies the concepts of internal and external stability. Nonlinear small-gain theorems for interconnections of ISS subsystems were established and extended to the ISpS (input-to-state practical stability) context in [5]; and a Lyapunov-based formulation was introduced in [6]. Summaries of various nonlinear small-gain theorems can be found in [7], [8].

On stabilizing interconnected control systems, in [5] a gain-assignment scheme was proposed to render feedback controls so that the small-gain condition holds in closed-loop. Similar techniques were employed in [9] for nonlinear cascaded systems with dynamic uncertainties, and in [10] for nonlinear feedforward systems with input unmodeled dynamics. See [8, Sec. 2.3] for an overview of the small-gain control design.

In this paper, we study the stabilization of interconnected switched systems. Switched systems have become a popular topic in recent years (see, e.g., [11] and references therein). It is well-known that, in general, a switched system does not inherit stability properties of the individual modes. For example, a switched system with two asymptotically stable modes may not even be stable [11, Part II]. In [12] it was proved that such a switched linear system is asymptotically stable provided that the switching admits a large enough dwell-time. This approach was generalized to the context of switched nonlinear systems and to the notion of average dwell-time (ADT) in [13]. A similar result was developed for switched linear systems with both stable and unstable modes in [14] by restricting in proportion the active time for unstable modes. Stability analysis of switched nonlinear systems was extended to the ISS context in [15], and to the IOSS (input/output-to-state stability) context in [16], which also considered non-IOSS modes. The stabilization of switched systems in the strict-feedback form was studied in [17]. In [18] a small-gain theorem for interconnected switched nonlinear systems with both ISS and non-ISS modes were established, assuming that the switching is slow in the sense of ADT and the active time of non-ISS modes is short in proportion. However, in [18] the Lyapunov gains of both switched systems were increased due to the switching and the non-ISS modes, making the small-gain condition more restrictive than the one for the case without switching.

Motivated by this undesirable effect, we study the feedback stabilization of interconnected switched control-affine systems with both ISS and non-ISS modes. ISpS of the interconnection with an arbitrarily small constant is achieved by establishing a Lyapunov-based small-gain theorem, and proposing a Lyapunov-based gain-assignment scheme. More specifically, we first extend the small-gain theorem [18, Th. 1] to the case of switched subsystems with both ISpS and non-ISpS modes, which allows us to select suitable gains and constants in the ISpS conditions on subsystems, for each arbitrarily small but fixed constant in the ISpS estimate for the interconnection. Then a Lyapunov-based gain-assignment approach inspired by [9] is developed to derive feedback controls for the required ISpS conditions.

In establishing the small-gain theorem for ISpS of interconnections of switched systems, we adopt various hybrid system techniques. Hybrid systems are dynamic systems exhibiting both continuous and discrete behaviors. Trajectory-based small-gain theorems for interconnections of hybrid systems were first reported in [19], [20], while Lyapunov-based formulations were introduced in [21]. In this work, we follow the modeling framework for hybrid systems in [22], which proved to be general and natural from the viewpoint of Lyapunov stability theory [23], [24]. Results on Lyapunov-based small-gain theorems using this modeling framework can be found in [25], [26], [27].

This paper is structured as follows. In Section II we introduce the system and stability notions. The problem formulation and the main result are presented in Section III.

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The main result is proved based on the small-gain theorem for ISpS of interconnections of switched systems in Section IV-A and the Lyapuov-based gain-assignment scheme in Section IV-B. Section V concludes the paper with a brief summary and an outlook on future research.

II. Preliminaries

Consider a family of dynamical systems with the state $x \in \mathbb{R}^n$, disturbance $w \in \mathbb{R}^m$ and *index set* $\mathcal{P}$ (which can in principle be arbitrary) modeled by

$$\dot{x} = f_p(x, w), \qquad p \in \mathcal{P}. \tag{1}$$

The corresponding *switched system* is defined by

$$\dot{x} = f_\sigma(x, w), \qquad x(0) = x_0 \tag{2}$$

with a piecewise constant, right-continuous *switching signal* $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ that specifies the active mode $\sigma(t)$ at each time t. For each $p \in \mathcal{P}$, the function f_p is locally Lipschitz and satisfies $f_p(0, 0) = 0$. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation (2) away from discontinuities of σ (in particular, there is no state jump). An admissible disturbance $w(\cdot)$ is a Lebesgue measurable, locally essentially bounded function. Discontinuities of σ are called *switching times*, or simply *switches*. It is assumed that the set of switches contains no accumulation points (thus there is at most one switch at any time and finitely many switches on any finite time interval).

We say that the switching signal σ admits a *dwell-time* [12] $\tau_d > 0$ if all consecutive switches t', t'' satisfy

$$t'' - t' \geq \tau_d; \tag{3}$$

and σ admits an *average dwell-time* (ADT) [13] $\tau_a > 0$ if

$$N(t_2, t_1) \leq \frac{t_2 - t_1}{\tau_a} + N_0 \qquad \forall\, t_2 > t_1 \geq 0 \tag{4}$$

with an integer $N_0 \geq 1$, where $N(t_2, t_1)$ denotes the number of switches on a time interval $(t_1, t_2]$. Note that (3) can be rewritten in the form of (4) with $\tau_a = \tau_d$ and $N_0 = 1$.

For two vectors v_1, v_2, let $(v_1, v_2) := (v_1^\top, v_2^\top)^\top$ denote their concatenation. For a vector v, let $|v|$ denote its Euclidean norm, and $|v|_\mathcal{A}$ its Euclidean distance to a set $\mathcal{A}$. For a function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, let $\|w\|$ denote its essential supremum Euclidean norm.

Let $\mathcal{C}^1$ denote the class of continuously differentiable functions. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing and positive definite. It is of *class* $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$ (in particular, this implies that it is globally invertible). A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t, and $\beta(r, \cdot)$ is continuous, strictly decreasing and $\lim_{t\to\infty} \beta(r, t) = 0$ for each fixed $r > 0$.

A system in (1) is *input-to-state practically stable (ISpS)* [5] if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and $\varepsilon \geq 0$ such that

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|w\|) + \varepsilon \qquad \forall\, t \geq 0 \tag{5}$$

for all initial states x_0 and admissible disturbances w. When $\varepsilon = 0$, ISpS becomes *input-to-state stability (ISS)* [4], which is equivalent to the standard notion of *global asymptotic stability (GAS)* for the case without disturbance [28, Prop. 2.5]. The same definitions of ISpS, ISS and GAS also apply to the switched system (2).

III. Main result

Consider an interconnection of two switched subsystems Σ_i (each with the state $x_i \in \mathbb{R}^{n_i}$, index set $\mathcal{P}_i$ and switching signal σ_i) for $i = 1, 2$ modeled by

$$\Sigma_i : \dot{x}_i = f_{i,\sigma_i}(x, w), \qquad x_i(0) = x_{i,0}, \tag{6}$$

where $x = (x_1, x_2)$ denotes the state of the interconnection, and $w \in \mathbb{R}^m$ the *external disturbance*. Each subsystem Σ_i switches independently and treats the state x_j of the other one as the *internal disturbance*.[1] We are interested in the scenario that both subsystems are in the control-affine form

$$\Sigma_i : \dot{x}_i = f^0_{i,\sigma_i}(x, w) + G_{i,\sigma_i}(x, w) u_i, \quad x_i(0) = x_{i,0} \tag{7}$$

with the control $u_i \in \mathbb{R}^{n_i}$. For each $p_i \in \mathcal{P}_i$, the open-loop dynamics f^0_{i,p_i} fulfills the same assumption as those imposed on f_p in Section II, and the matrix-valued function G_{i,p_i} is locally Lipschitz. An admissible feedback control is of the form $u_i = \kappa_{i,\sigma_i}(x_i)$ with a family of positive definite, locally Lipschitz functions κ_{i,p_i} for $p_i \in \mathcal{P}_i$. Our goal is to construct suitable feedback controls u_1, u_2 such that (7) is ISpS (w.r.t. the external disturbance w) with an arbitrarily small $\varepsilon > 0$.

We consider the general scenario where both open-loop subsystems in (7) contain ISS and non-ISS modes. Let $\mathcal{P}_{s,i}$ and $\mathcal{P}_{u,i}$ denote the index sets of ISS and non-ISS modes, respectively. Then $(\mathcal{P}_{s,i}, \mathcal{P}_{u,i})$ forms a partition of $\mathcal{P}_i$ (i.e., $\mathcal{P}_{s,i} \cup \mathcal{P}_{u,i} = \mathcal{P}_i$ and $\mathcal{P}_{s,i} \cap \mathcal{P}_{u,i} = \emptyset$). Following [16], we let $T_{s,i}(t_2, t_1)$ denote the total active time of ISS modes on a time interval $(t_1, t_2]$, and $T_{u,i}(t_2, t_1)$ that of non-ISS modes. Then $T_{s,i}(t_2, t_1) + T_{u,i}(t_2, t_1) = t_2 - t_1$.

Our first assumption is that each ISS mode admits an ISS-Lyapunov function, each non-ISS mode admits a candidate ISS-Lyapunov function, and the (candidate) ISS-Lyapunov functions are uniform in the following sense.

Assumption 1 (Generalized ISS-Lyapunov)**.** For the subsystem Σ_i of (7) in open-loop, there exists a family of positive definite and $\mathcal{C}^1$ functions $V_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ for $p_i \in \mathcal{P}_i$ such that their gradients $\nabla V_{i,p_i}$ are locally Lipschitz and nowhere vanishing except at the origin, and that

1. there exist bounds $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$ such that
$$\alpha_{1,i}(|x_i|) \leq V_{i,p}(x) \leq \alpha_{2,i}(|x_i|) \quad \forall\, x_i, \forall\, p \in \mathcal{P}_i; \tag{8}$$
2. there exist internal gain $\phi_i \in \mathcal{K}_\infty$, external gain $\chi^w_i \in \mathcal{K}_\infty$ and rate coefficients $\lambda_{s,i}, \lambda_{u,i} > 0$ such that
$$|x_i| \geq \chi^w_i(|w|) \Rightarrow \begin{cases} \nabla V_{i,p_s}(x_i) \cdot f^0_{i,p_s}(x, w) \leq -\lambda_{s,i} V_{i,p_s}(x_i) + \phi_i(|x_j|), \\ \nabla V_{i,p_u}(x_i) \cdot f^0_{i,p_u}(x, w) \leq \lambda_{u,i} V_{i,p_u}(x_i) + \phi_i(|x_j|) \end{cases} \tag{9}$$

[1] Throughout this paper, we always follow the convention that $i \in \{1, 2\}$ and $j \in \{1, 2\} \backslash \{i\}$.

for all x_i, x_j, w and all $p_s \in \mathcal{P}_{s,i}$, $p_u \in \mathcal{P}_{u,i}$;

3. there exists a ratio $\mu_i \geq 1$ such that

$$V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i) \qquad \forall x_i, \forall p, q \in \mathcal{P}_i. \tag{10}$$

Remark 1. For each $p_s \in \mathcal{P}_{s,i}$ (ISS mode), the assumption that $\nabla V_{i,p_s}(x_i) \neq 0$ when $x_i \neq 0$ is guaranteed by the first inequality in (9), which cannot hold with $x_i \neq 0$, $x_j = 0$, $w = 0$ and $\nabla V_{i,p_s}(x_i) = 0$.

Also, we assume that the switching is slow in the sense of ADT, and the active time of non-ISS modes is short in proportion.

Assumption 2 (ADT)**.** The switching signal σ_i satisfies (4) with an ADT $\tau_{a,i} > 0$ and an integer $N_{0,i} \geq 1$.

Assumption 3 (Time-ratio)**.** There exist a *time-ratio* $\rho_i \in [0,1)$ and a constant $T_{0,i} \geq 0$ such that

$$T_{u,i}(t_2,t_1) \leq T_{0,i} + \rho_i(t_2 - t_1) \qquad \forall t_2 > t_1 \geq 0.$$

Finally, the matrix-valued functions $G_{i,p_i} : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^{n_i \times n_i}$ for $p_i \in \mathcal{P}_i$ satisfy the following condition.

Assumption 4. For each $p_i \in \mathcal{P}_i$,

$$G_{i,p_i}(x,w) + G_{i,p_i}(x,w)^\top - 2\varepsilon^G_{i,p_i} I \geq 0 \qquad \forall x, \forall w \tag{11}$$

(i.e., the matrix on the left-hand side is positive semi-definite) with a constant $\varepsilon^G_{i,p_i} > 0$.

This assumption ensures that it does not require arbitrarily large controls to achieve stabilization.[2] Similar assumptions can be seen in the literature such as [9, Assumptions 5 and 9].

Our main result is stated as the following theorem:

Theorem 1. *Consider the interconnection* (7)*. Suppose that for each subsystem* Σ_i*, Assumptions 1–4 hold with*

$$(1-\rho_i)\lambda_{s,i} - \rho_i\lambda_{u,i} - (\ln\mu_i)/\tau_{a,i} > 0. \tag{12}$$

Then for each $\varepsilon > 0$*, there exist suitable feedback controls* u_1, u_2 *such that* (7) *is ISpS with the constant* ε*, that is, there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *such that* (5) *holds.*

IV. Proof of the Main Result

A. A Lyapunov-based small-gain theorem

First, we extend [18, Th. 1] to establish a small-gain theorem for ISpS of the interconnection (6).

Assumption 1′ (Generalized ISpS-Lyapunov)**.** For the subsystem Σ_i of (6), there exists a family of positive definite and $\mathcal{C}^1$ functions $V_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ for $p_i \in \mathcal{P}_i$ such that

1. there exist bounds $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$ such that (8) holds;
2. there exist internal gain $\chi_i \in \mathcal{K}_\infty$, external gain $\chi^w_i \in \mathcal{K}_\infty$, constant $\delta_i > 0$ and rate coefficients $\lambda_{s,i}, \lambda_{u,i} > 0$ such that

$$|x_i| \geq \max\{\chi_i(|x_j|), \chi^w_i(|w|), \delta_i\} \Rightarrow \begin{cases} \nabla V_{i,p_s}(x_i) \cdot f_{i,p_s}(x,w) \leq -\lambda_{s,i} V_{i,p_s}(x_i), \\ \nabla V_{i,p_u}(x_i) \cdot f_{i,p_u}(x,w) \leq \lambda_{u,i} V_{i,p_u}(x_i) \end{cases} \tag{13}$$

for all x_i, x_j, w and all $p_s \in \mathcal{P}_{s,i}$, $p_u \in \mathcal{P}_{u,i}$;

3. there exists a ratio $\mu_i \geq 1$ such that (10) holds.

The following proposition provides the small-gain condition for ISpS of the interconnection (6), and the relation between ε in (5) and δ_i in (13) for $i = 1, 2$.

Proposition 2. *Consider the interconnection* (6)*. Suppose that for each* Σ_i*, Assumptions 1′, 2, 3 and* (12) *hold. Let*

$$\Theta_i := N_{0,i}\ln\mu_i + T_{0,i}(\lambda_{s,i} + \lambda_{u,i}), \qquad i = 1,2 \tag{14}$$

and consider the Lyapunov gains $\psi_1, \psi_2 \in \mathcal{K}_\infty$ *defined by*

$$\psi_i(r) := \alpha_{2,i}(\chi_i(\alpha^{-1}_{1,j}(r)))e^{\Theta_i}, \qquad i = 1,2. \tag{15}$$

Then (6) *is ISpS if* ψ_1, ψ_2 *satisfy the small-gain condition*

$$\psi_1(\psi_2(r)) < r \qquad \forall r > 0. \tag{16}$$

In particular, (5) *holds for all constants* ε *satisfying*

$$\varepsilon \geq \sqrt{2}\max\big\{\alpha^{-1}_{1,1}(\alpha_{2,1}(\delta_1)e^{\Theta_1}),\ \chi^{-1}_1(\delta_1),\ \alpha^{-1}_{1,2}(\alpha_{2,2}(\delta_2)e^{\Theta_2}),\ \chi^{-1}_2(\delta_2)\big\}. \tag{17}$$

Proof. See Appendix I for the proof of Proposition 2. □

B. Gain assignment

Next, we extend the techniques in [9] (cf. [8, Sec. 2.3]) to develop a Lyapunov-based gain-assignment scheme that renders suitable feedback controls for each fixed internal gain and constant. While both methods require partial knowledge of the dynamics, apart from being developed for switched systems, ours is different in the sense that we assume knowledge of the gradients of the ISS-Lyapunov functions instead of the $\mathcal{K}_\infty$ bounds of the dynamics as in [9].

Proposition 3. *Consider the subsystem* Σ_i *in* (7)*. Suppose that Assumptions 1, 4 hold. Given arbitrary* $\chi_i \in \mathcal{K}_\infty$ *and* $\delta_i > 0$*, there exists a feedback control* $u_i = \kappa_{i,\sigma_i}(x_i)$ *(given in* (34) *below) such that* (13) *holds in closed-loop.*

Proof. See Appendix II for the proof of Proposition 3. □

C. Control synthesis

Now we combine the results above to prove Theorem 1.

First, select $\chi_1, \chi_2 \in \mathcal{K}_\infty$ such that (16) holds with ψ_1, ψ_2 defined by (15).

Second, given an arbitrary $\varepsilon > 0$, select small enough $\delta_1, \delta_2 > 0$ so that (17) holds, such as

$$\delta_i = \min\big\{\alpha^{-1}_{2,i}(\alpha_{1,i}(\varepsilon/\sqrt{2})/e^{\Theta_i}),\ \chi_i(\varepsilon/\sqrt{2})\big\}, \quad i = 1,2.$$

Finally, for each subsystem Σ_i in (7), invoke Proposition 3 to obtain the suitable feedback control u_i such that (13), and hence Assumption 1′, holds in closed-loop. Then from Proposition 2 it follows that (7) is ISpS with the constant ε.

V. Conclusion

We studied the stabilization of interconnected switched control-affine systems with both ISS and non-ISS modes. Based on a small-gain theorem and a Lyapunov-based gain-assignment scheme, suitable feedback controls were designed to achieve ISpS with an arbitrarily small constant. Future work will focus on extending the results to more general types of interconnections.

[2] Our result also applies to the case that $G_{i,p_i} + G^\top_{i,p_i} \leq -2\varepsilon^G_{i,p_i} < 0$ everywhere, by changing the signs of the feedback controls in (34) below.

Appendix I
Proof of Proposition 2

A. Preliminaries for hybrid systems

Following [24], a hybrid system with the state $x \in \mathcal{X} \subset \mathbb{R}^n$ and input $w \in \mathcal{W} \subset \mathbb{R}^m$ is modeled by

$$\begin{aligned} \dot{x} &\in F(x, w), \qquad && (x, w) \in \mathcal{C}, \\ x^+ &\in G(x, w), \qquad && (x, w) \in \mathcal{D}. \end{aligned} \tag{18}$$

We call $\mathcal{C} \subset \mathcal{X} \times \mathcal{W}$ the *flow set*, $\mathcal{D} \subset \mathcal{X} \times \mathcal{W}$ the *jump set*, $F : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathbb{R}^n$ the *flow map*[3], and $G : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathcal{X}$ the *jump map*. In this model, the state x follows the continuous flow if $(x, u) \in \mathcal{C} \backslash \mathcal{D}$, and the discrete jump if $(x, u) \in \mathcal{D} \backslash \mathcal{C}$. If $(x, u) \in \mathcal{C} \cap \mathcal{D}$ then it may either flow or jump. A solution of (18) is defined on a *hybrid time domain* $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, which is a union of a finite or infinite sequence of intervals $[t_k, t_{k+1}] \times \{k\}$, with the last one (if existent) possibly of the form $[t_k, T) \times \{k\}$ with $T \in \mathbb{R}$ or $T = \infty$. A *hybrid input* is a function $w : \operatorname{dom} w \to \mathcal{W}$ defined on a hybrid time domain such that $w(\cdot, k)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t, k) \in \operatorname{dom} w\}$ for each fixed k. A *solution* $x : \operatorname{dom} x \to \mathcal{X}$ of (19) with a hybrid input $w : \operatorname{dom} w \to \mathcal{W}$ satisfies that $x(\cdot, k)$ is locally absolutely continuous on $\{t : (t, k) \in \operatorname{dom} x\}$ for each fixed k, and[4]

- $\operatorname{dom} x = \operatorname{dom} w$;
- $(x(t, k), w(t, k)) \in \mathcal{C}$ and $\dot{x}(t, k) \in F(x(t, k), w(t, k))$ for all k and almost all t such that $(t, k) \in \operatorname{dom} x$;
- $(x(t, k), w(t, k)) \in \mathcal{D}$ and $x(t, k+1) \in G(x(t, k), w(t, k))$ for all $(t, k) \in \operatorname{dom} x$ such that $(t, k + 1) \in \operatorname{dom} x$.

With suitable assumptions on $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$, one can establish local existence of solutions, which are not necessarily unique (see, e.g., [22, Sec. 2]). A solution is *maximal* if it cannot be extended, and *complete* if its domain is unbounded.

For a hybrid input $w : \operatorname{dom} w \to \mathbb{R}^m$, its essential supremum Euclidean norm is defined by

$$\|w\| := \max\left\{ \operatorname*{ess\,sup}_{(s,l) \in \operatorname{dom} w} |w(s, l)|, \sup_{(s,l) \in \mathcal{J}(w)} |w(s, l)| \right\},$$

where $\mathcal{J}(w) := \{(s, l) \in \operatorname{dom} w : (s, l + 1) \in \operatorname{dom} w\}$.[5]

For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, its *Clarke derivative* [29] at x in the direction $v \in \mathbb{R}^n$ is defined by

$$V^\circ(x; v) := \limsup_{s \searrow 0,\, y \to x} \frac{V(y + sv) - V(y)}{s}.$$

B. Auxiliary timers and hybrid systems

We augment each switched subsystem Σ_i in (6) with an auxiliary timer incorporating the conditions on switching to obtain a corresponding hybrid system. Consider a hybrid system with the state $z_i = (\tilde{x}_i, \tilde{\sigma}_i, \tau_i) \in \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Theta_i] =: \mathcal{Z}_i$ and the input $d_i = (\tilde{v}_i, \tilde{w}_i) \in \mathbb{R}^{n_j} \times \mathbb{R}^m$ modeled by

$$\begin{aligned} \dot{z}_i &\in F_i(z_i, d_i), \qquad && (z_i, d_i) \in \mathcal{C}_i, \\ z_i^+ &\in G_i(z_i), \qquad && (z_i, d_i) \in \mathcal{D}_i \end{aligned} \tag{19}$$

with

$$F_i(z_i, d_i) := \begin{cases} \begin{bmatrix} \{f_{i,\tilde{\sigma}_i}(\tilde{x}_i, d_i)\} \\ \{0\} \\ [0, \theta_i] \end{bmatrix}, & \tilde{\sigma}_i \in \mathcal{P}_{s,i}; \\ \begin{bmatrix} \{f_{i,\tilde{\sigma}_i}(\tilde{x}_i, d_i)\} \\ \{0\} \\ \{\theta_i - (\lambda_{s,i} + \lambda_{u,i})\} \end{bmatrix}, & \tilde{\sigma}_i \in \mathcal{P}_{u,i}, \end{cases}$$

$$\begin{aligned} \mathcal{C}_i &:= \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Theta_i] \times \mathbb{R}^{n_j} \times \mathbb{R}^m, \\ G_i(z_i) &:= \{\tilde{x}_i\} \times (\mathcal{P}_i \backslash \{\tilde{\sigma}_i\}) \times \{\tau_i - \ln \mu_i\}, \\ \mathcal{D}_i &:= \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln \mu_i, \Theta_i] \times \mathbb{R}^{n_j} \times \mathbb{R}^m, \end{aligned}$$

where Θ_i is defined by (14) and

$$\theta_i := (\ln \mu_i)/\tau_{a,i} + \rho_i(\lambda_{s,i} + \lambda_{u,i}) < \lambda_{s,i}. \tag{20}$$

Note the inequality in (20) follows from (12). The following lemma characterizes the correspondence between solutions of Σ_i in (6) and complete solutions of (19).

Lemma 1. *Let x_i be a solution of the switched subsystem Σ_i in* (6) *with the internal disturbance x_j, external disturbance w and switching signal σ_i. Suppose that Assumptions 1′, 2, 3 and* (12) *hold. Then there exists a complete solution $z_i = (\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ of the hybrid system* (19) *with the hybrid input $d_i = (\tilde{v}_i, \tilde{w}_i)$ such that for all $(t, k) \in \operatorname{dom} z_i$,*

$$\tilde{v}_i(t, k) = x_j(t),\ \tilde{w}_i(t, k) = w(t),\ \tilde{x}_i(t, k) = x_i(t). \tag{21}$$

Proof. The proof is similar to that of [18, Prop. 1] and is omitted here. □

C. Hybrid ISpS-Lyapunov functions

Consider the function $V_i : \mathcal{Z}_i \to \mathbb{R}_{\geq 0}$ defined by

$$V_i(z_i) := V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i}$$

with the family of functions V_{i,p_i} for $p_i \in \mathcal{P}_i$ in Assumption 1′. As all V_{i,p_i} are $\mathcal{C}^1$, it follows that V_i is $\mathcal{C}^1$ in $\tilde{x}_i$ and τ_i. Moreover, it satisfies the following conditions.

Lemma 2. *Suppose Assumptions 1′, 2, 3 and* (12) *hold. Then*

1. *for the bounds $\tilde{\alpha}_{1,i}, \tilde{\alpha}_{2,i} \in \mathcal{K}_\infty$ defined by*

$$\tilde{\alpha}_{1,i}(r) := \alpha_{1,i}(r), \qquad \tilde{\alpha}_{2,i}(r) := \alpha_{2,i}(r) e^{\Theta_i}$$

and the set $\mathcal{A}_i := \{0\} \times \mathcal{P}_i \times [0, \Theta_i] \subset \mathcal{Z}_i$, it holds that

$$\tilde{\alpha}_{1,i}(|z_i|_{\mathcal{A}_i}) \leq V_i(z_i) \leq \tilde{\alpha}_{2,i}(|z_i|_{\mathcal{A}_i}) \qquad \forall\, z_i \in \mathcal{Z}_i;$$

2. *for the rate coefficient $\lambda_i > 0$ defined by $\lambda_i := \lambda_{s,i} - \theta_i$, it holds that for all $(z_i, \tilde{v}_i, \tilde{w}_i) \in \mathcal{C}_i$ and $v_i \in F_i(z_i, \tilde{v}_i, \tilde{w}_i)$,*

$$\begin{aligned} &|z_i|_{\mathcal{A}_i} \geq \max\{\chi_i(|\tilde{v}_i|),\, \chi_i^w(|\tilde{w}_i|),\, \delta_i\} \\ &\Rightarrow \nabla V_i(z_i) \cdot v_i \leq -\lambda_i V_i(z_i); \end{aligned} \tag{22}$$

3. *it holds that*

$$V_i(z_i^+) \leq V_i(z_i) \qquad \forall\, (z_i, \tilde{v}_i, \tilde{w}_i) \in \mathcal{D}_i,\, \forall\, z_i^+ \in G_i(z_i).$$

Proof. The proof of is similar to that of [18, Prop. 2] and is omitted here. □

[3] We use "$\rightrightarrows$" to denote a set-valued mapping.

[4] Here $x(t, k)$ represents the state of (19) at time t and after k jumps.

[5] Note that the set of hybrid jump times $\mathcal{J}(w)$ with measure 0 cannot be ignored when computing the essential supremum norm.

D. ISpS of the interconnection

Following [6, Lemma A.1], if (16) holds then there exists a gain $\psi \in \mathcal{K}_\infty$ such that $\psi \in \mathcal{C}^1$ with $\psi' > 0$ on $\mathbb{R}_{>0}$ and

$$\psi_1^{-1}(r) > \psi(r) > \psi_2(r) \qquad \forall\, r > 0. \tag{23}$$

Let $z = (z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2 =: \mathcal{Z}$ and consider the function $V : \mathcal{Z} \to \mathbb{R}_{\geq 0}$ defined by

$$V(z) := \max\{\psi(V_1(z_1)),\, V_2(z_2)\}.$$

As both V_i are $\mathcal{C}^1$ in $\tilde{x}_i$ and τ_i, and $\psi \in \mathcal{C}^1$ on $R_{>0}$, it follows that V is locally Lipschitz and hence absolutely continuous and almost everywhere differentiable away from its zero set (Rademacher's theorem [30]). Moreover, based on Lemma 2 it satisfies the following conditions.

Lemma 3. *Suppose Assumptions 1′, 2, 3 and* (12) *hold. Then*
1. *for the bounds $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ defined by*

$$\begin{aligned} \alpha_1(r) &:= \min\{\psi(\tilde{\alpha}_{1,1}(r/\sqrt{2})),\, \tilde{\alpha}_{1,2}(r/\sqrt{2})\}, \\ \alpha_2(r) &:= \max\{\psi(\tilde{\alpha}_{2,1}(r)),\, \tilde{\alpha}_{2,2}(r)\} \end{aligned} \tag{24}$$

and the set $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2$, it holds that

$$\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}}) \qquad \forall\, z \in \mathcal{Z}; \tag{25}$$

2. *for the gain $\chi^w \in \mathcal{K}_\infty$, constant $\delta > 0$ and positive definite, continuous rate $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by*

$$\begin{aligned} \chi^w(r) &:= \max\{\psi(\tilde{\alpha}_{2,1}(\chi_1^w(r))),\, \tilde{\alpha}_{2,2}(\chi_2^w(r))\}, \\ \delta &:= \max\{\psi(\tilde{\alpha}_{2,1}(\delta_1)),\, \tilde{\alpha}_{2,2}(\delta_2)\}, \\ h(r) &:= \min\{\psi'(\psi^{-1}(r))\lambda_1\psi^{-1}(r),\, \lambda_2 r\}, \end{aligned} \tag{26}$$

it holds that for all $z \in \mathcal{Z}$, $\tilde{w} \in \mathbb{R}^m$ and $v \in F_1(z_1, \tilde{x}_2, \tilde{w}) \times F_2(z_2, \tilde{x}_1, \tilde{w})$,

$$V(z) \geq \max\{\chi^w(|\tilde{w}|),\, \delta\} \Rightarrow V^\circ(z; v) \leq -h(V(z)); \tag{27}$$

3. *it holds that for all $z \in \mathbb{R}^{n_1} \times \mathcal{P}_1 \times [\ln\mu_1, \Theta_1] \times \mathbb{R}^{n_2} \times \mathcal{P}_2 \times [\ln\mu_2, \Theta_2]$ and $z^+ \in G_1(z_1) \times G_2(z_2)$, or $z \in \mathcal{Z}_1 \times \mathbb{R}^{n_2} \times \mathcal{P}_2 \times [\ln\mu_2, \Theta_2]$ and $z^+ \in \{z_1\} \times G_2(z_2)$, or $z \in \mathbb{R}^{n_1} \times \mathcal{P}_1 \times [\ln\mu_1, \Theta_1] \times \mathcal{Z}_2$ and $z^+ \in G_1(z_1) \times \{z_2\}$,*

$$V(z^+) \leq V(z). \tag{28}$$

Proof. The proof of item 1 and 3 are the same as that of the corresponding conditions in [18, Sec. 4.5] and are omitted here. For item 2, let $v = (v_1, v_2)$ be such that $v_i \in F_i(z_i, \tilde{x}_j, \tilde{w})$ and consider the following three cases:
1. If $\psi(V_1(z_1)) > V_2(z_2)$ then $V(z) = \psi(V_1(z_1))$. Hence

$$|z_1|_{\mathcal{A}_1} \geq \tilde{\alpha}_{2,1}^{-1}(\psi_1(\tilde{\alpha}_{1,2}(|z_2|_{\mathcal{A}_2}))) = \chi_1(|z_2|_{\mathcal{A}_2}) \tag{29}$$

following (15). If $V(z) \geq \max\{\chi^w(|\tilde{w}|),\, \delta\}$ then

$$|z_1|_{\mathcal{A}_1} \geq \tilde{\alpha}_{2,1}^{-1}(V_1(z_1)) \geq \max\{\chi_1^w(|\tilde{w}|),\, \delta_1\}. \tag{30}$$

Therefore from (22) with $i = 1$ and (26) it follows that

$$V^\circ(z; v) = \psi'(V_1(z_1))\nabla V_1(z_1) \cdot v_1 \leq -h(V(z)).$$

2. If $\psi(V_1(z_1)) < V_2(z_2)$ then $V(z) = V_2(z_2)$. Hence

$$|z_2|_{\mathcal{A}_2} \geq \tilde{\alpha}_{2,2}^{-1}(\psi_2(\tilde{\alpha}_{1,1}(|z_1|_{\mathcal{A}_1}))) = \chi_2(|z_1|_{\mathcal{A}_1}) \tag{31}$$

following (15). If $V(z) \geq \max\{\chi^w(|\tilde{w}|),\, \delta\}$ then

$$|z_2|_{\mathcal{A}_2} \geq \tilde{\alpha}_{2,2}^{-1}(V_2(z_2)) \geq \max\{\chi_2^w(|\tilde{w}|),\, \delta_2\}. \tag{32}$$

Therefore from (22) with $i = 2$ and (26) it follows that

$$V^\circ(z; v) = \nabla V_2(z_2) \cdot v_2 \leq -\lambda_2 V_2(z_2) \leq -h(V(z)).$$

3. Otherwise $V(z) = \psi(V_1(z_1)) = V_2(z_2)$. Then $V(z) \geq \max\{\chi^w(|\tilde{w}|),\, \delta\}$ implies that (29)–(32) all hold. By virtue of [27, Lemma II.1], $V^\circ(z; v)$ is well-defined, and from the proof of the first two cases it follows that

$$\begin{aligned} V^\circ(z; v) &\leq \max\{\psi'(V_1(z_1))\nabla V_1(z_1) \cdot v_1, \nabla V_2(z_2) \cdot v_2\} \\ &\leq -h(V(z)). \end{aligned}$$

□

Let $x = (x_1, x_2)$ be a solution of the interconnection (6) with the external disturbance w and switching signals σ_1, σ_2. Following Lemma 1, for each i there exists a complete solution $\bar{z}_i = (\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ of the hybrid system (19) with the hybrid input $d_i = (\tilde{v}_i, \tilde{w}_i)$ such that (21) holds for all $(t, k) \in \operatorname{dom} \bar{z}_i$. As σ_1, σ_2 are independent, $\operatorname{dom} \bar{z}_1, \operatorname{dom} \bar{z}_2$ are different in general. Define a hybrid time domain E so that for each $(t, k) \in E$, $(t, k{+}1) \in E$ if and only if there are $i \in \{1, 2\}$ and $k_i \in \mathbb{Z}_{\geq 0}$ so that $(t, k_i), (t, k_i + 1) \in \operatorname{dom} \bar{z}_i$. Define $z = (z_1, z_2) : E \to \mathcal{Z}$ as follows: for each $(t, k) \in E$,
1. when $(t, k-1), (t, k+1) \notin E$, for each i let $z_i(t, k) = \bar{z}_i(t, k_i)$ for the unique k_i such that $(t, k_i) \in \operatorname{dom} \bar{z}_i$;
2. when $(t, k+1) \in E$, for each i if there is a k_i such that $(t, k_i), (t, k_i{+}1) \in \operatorname{dom} \bar{z}_i$ then let $z_i(t, k) = \bar{z}_i(t, k_i)$ and $z_i(t, k{+}1) = \bar{z}_i(t, k_i{+}1)$; else let $z_i(t, k) = z_i(t, k{+}1) = \bar{z}_i(t, k_i)$ for the unique k_i such that $(t, k_i) \in \operatorname{dom} \bar{z}_i$.

Define $\tilde{w} : E \to \mathbb{R}^m$ from $\tilde{w}_1, \tilde{w}_2$ in a similar manner. Then

$$|z(t, k)|_{\mathcal{A}} = |x(t)|,\ \tilde{w}(t, k) = w(t) \quad \forall\, (t, k) \in E. \tag{33}$$

Consider the hybrid time (t_0, k_0) defined by

$$(t_0, k_0) := \operatorname*{argmin}_{(s,l) \in E}\{s{+}l : V(z(s, l)) \leq \max\{\chi^w(\|\tilde{w}\|),\, \delta\}\}.$$

From (27) and (28) it follows that

$$V(z(t, k)) \leq V(z(t_0, k_0)) \leq \max\{\chi^w(\|\tilde{w}\|),\, \delta\}$$

for all $(t, k) \in E$ with $t + k > t_0 + k_0$. Moreover, following similar arguments as in [18, Sec. 4.5], (27) and (28) also imply that there exists a function $\beta_V \in \mathcal{KL}$ such that

$$V(z(t, k)) \leq \beta_V(V(z(0, 0)), t)$$

for all $(t, k) \in E$ with $t{+}k \leq t_0{+}k_0$. Then for all $(t, k) \in E$,

$$V(z(t, k)) \leq \max\{\beta_V(V(z(0, 0)), t),\, \chi^w(\|\tilde{w}\|),\, \delta\}.$$

Following (25) and (33), the solution x of the interconnection (6) satisfies (5) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ defined by

$$\beta(r, t) := \alpha_1^{-1}(\beta_V(\alpha_2(r), t)), \quad \gamma(r) := \alpha_1^{-1}(\chi^w(r)),$$

and for all $\varepsilon \geq \alpha_1^{-1}(\delta)$. We conclude the proof of Proposition 2 by noting that (17) implies $\varepsilon \geq \alpha_1^{-1}(\delta)$. More specifically,

$$\begin{aligned}\alpha_1^{-1}(\delta) &\le \sqrt{2}\max\big\{\tilde\alpha_{1,2}^{-1}(\psi(\tilde\alpha_{2,1}(\delta_1))),\, \tilde\alpha_{1,2}^{-1}(\tilde\alpha_{2,2}(\delta_2)),\\ &\qquad \tilde\alpha_{1,1}^{-1}(\psi^{-1}(\psi(\tilde\alpha_{2,1}(\delta_1)))),\, \tilde\alpha_{1,1}^{-1}(\psi^{-1}(\tilde\alpha_{2,2}(\delta_2)))\big\}\\ &\le \sqrt{2}\max\big\{\tilde\alpha_{1,1}^{-1}(\tilde\alpha_{2,1}(\delta_1)),\, \tilde\alpha_{1,2}^{-1}(\tilde\alpha_{2,2}(\delta_2)),\\ &\qquad \tilde\alpha_{1,2}^{-1}(\psi_1^{-1}(\tilde\alpha_{2,1}(\delta_1))),\, \tilde\alpha_{1,1}^{-1}(\psi_2^{-1}(\tilde\alpha_{2,2}(\delta_2)))\big\}\\ &= \sqrt{2}\max\big\{\alpha_{1,1}^{-1}(\alpha_{2,1}(\delta_1)e^{\Theta_1}),\, \chi_1^{-1}(\delta_1),\\ &\qquad \alpha_{1,2}^{-1}(\alpha_{2,2}(\delta_2)e^{\Theta_2}),\, \chi_2^{-1}(\delta_2)\big\},\end{aligned}$$

following (15), (23), (24) and (26).

Appendix II
Proof of Proposition 3

Let arbitrary $\chi_i \in \mathcal{K}_\infty$ and $\delta_i > 0$ be given and fixed. For each $p_i \in \mathcal{P}_i$, define the function $\xi_{i,p_i} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$\xi_{i,p_i}(r) := \begin{cases} \min\limits_{\delta_i \le |y| \le r} |\nabla V_{i,p_i}(y)|^2, & r > \delta_i;\\ \min\limits_{|y|=\delta_i} |\nabla V_{i,p_i}(y)|^2, & r \le \delta_i,\end{cases}$$

where V_{i,p_i} is as in Assumption 1. Then ξ_{i,p_i} is continuous, decreasing and (strictly) positive. Hence the function $\bar\nu_{i,p_i}$: $\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ defined by

$$\bar\nu_{i,p_i}(r) := \phi_i(\chi_i^{-1}(r))/\xi_{i,p_i}(r)$$

is of class $\mathcal{K}_\infty$. Following [9, Lemma 1], there exists a smooth function $\nu_{i,p_i} \in \mathcal{K}_\infty$ such that

$$\nu_{i,p_i}(r) \ge \bar\nu_{i,p_i}(r) \qquad \forall\, r \ge \delta_i.$$

Consider the feedback control $u_i = \kappa_{i,\sigma_i}(x_i)$ with the family of functions $\kappa_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_{\ge 0}$ for $p_i \in \mathcal{P}_i$ defined by

$$\kappa_{i,p_i}(x_i) := -\frac{\nu_{i,p_i}(|x_i|)}{\varepsilon^G_{i,p_i}}\nabla V_{i,p_i}(x_i), \tag{34}$$

with the constant ε^G_{i,p_i} in (11). If $|x_i| \ge \max\{\chi_i(|x_j|),\, \delta_i\}$ then

$$\begin{aligned}&\nabla V_{i,p_i}(x_i)\cdot G_{i,p_i}(x_i)u_i\\ &= -\frac{\nabla V_{i,p_i}(x_i)^\top G_{i,p_i}(x_i)\nabla V_{i,p_i}(x_i)}{\varepsilon^G_{i,p_i}}\nu_{i,p_i}(|x_i|)\\ &\le -\frac{\nabla V_{i,p_i}(x_i)^\top G_{i,p_i}(x_i)\nabla V_{i,p_i}(x_i)}{\varepsilon^G_{i,p_i}\min_{\delta_i \le |y| \le |x_i|}|\nabla V_{i,p_i}(y)|^2}\phi_i(\chi_i^{-1}(|x_i|))\\ &\le -\phi_i(|x_j|)\end{aligned}$$

for all $p_i \in \mathcal{P}_i$, where the last inequality follows partially from (11). Combining the previous implication with (9) yields (13), which concludes the proof of Proposition 3.

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