Stabilization of interconnected switched control-affine systems via a Lyapunov-based small-gain approach

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Switched system

- Switching is ubiquitous in realistic system models, such as
  - Thermostat
  - Gear tranmission
  - Power supply
- Structure of a switched system
  - A family of dynamics, called modes
  - A sequence of events, called switches
- In this work: time-dependent, uncontrolled switching

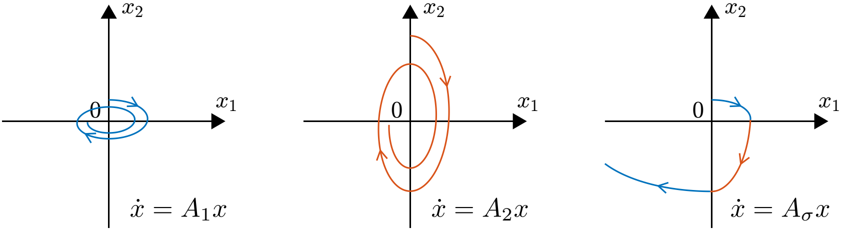
Presentation outline

- Preliminaries
- Interconnected switched systems and small-gain theorem
- Stabilization via a small-gain approach

Nonlinear switched system with input

$$\dot{x} = f_\sigma(x, w), \qquad x(0) = x_0$$

- State $x \in \mathbb{R}^n$, disturbance $w \in \mathbb{R}^m$
- A family of modes f_p, $p \in \mathcal{P}$, with an index set $\mathcal{P}$
- A right-continuous, piecewise constant switching signal $\sigma : \mathbb{R}_+ \to \mathcal{P}$ that indicates the active mode $\sigma(t)$
- Solution $x(\cdot)$ is absolutely continuous (no state jump)



Stability notions

Definition (GAS)

A continuous-time system is globally asymptotically stable (GAS) if there is a function $\beta \in \mathcal{KL}$ s.t. for all initial state x_0,

$$|x(t)| \leq \beta(|x_0|, t) \qquad \forall\, t \geq 0.$$

- A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, positive definite and strictly increasing; $\alpha \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if $\lim_{r\to\infty} \alpha(r) = \infty$, such as $\alpha(r) = r^2$ or $|r|$
- A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{t\to\infty} \gamma(t) = 0$, such as $\gamma(t) = e^{-t}$
- A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t, and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed $r > 0$, such as $\beta(r, t) = r^2 e^{-t}$

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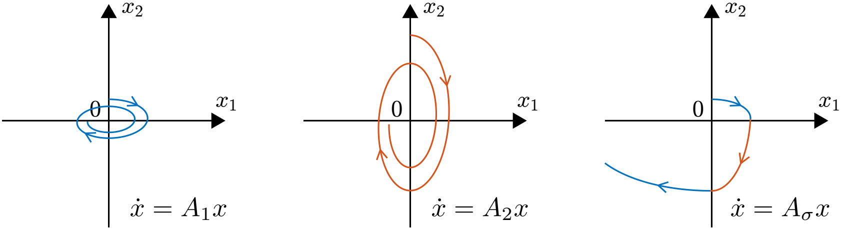
- The switched system may be unstable even if all individual modes are GAS



Stability notions

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Definition (ISpS [JTP94])

A continuous-time system is input-to-state practically stable (ISpS) if there are functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and a constant $\varepsilon \geq 0$ s.t. for all x_0 and disturbance w,

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|w\|) + \varepsilon \qquad \forall\, t \geq 0.$$

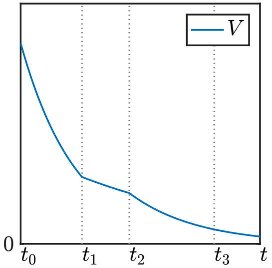
- When $\varepsilon = 0$, ISpS becomes input-to-state stability (ISS) [Son89]
- When $\varepsilon = 0$ and $\gamma \equiv 0$, ISpS becomes GAS

[JTP94] Z.-P. Jiang, A. R. Teel, and L. Praly, *Mathematics of Control, Signals, and Systems*, 1994
[Son89] E. D. Sontag, *IEEE Transactions on Automatic Control*, 1989

Lyapunov characterizations

$$\dot{x} = f_\sigma(x, w), \qquad x(0) = x_0$$

- A common Lyapunov function



The switched system is GAS if

- it admits a Lyapunov function V which decreases along the solution in all modes:

$$D_{f_p} V(x, w) \leq -\lambda V(x)$$

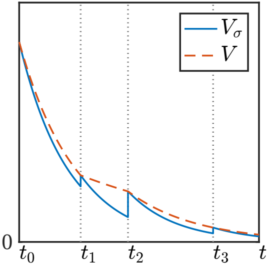
with a constant $\lambda > 0$.

[PW96] L. Praly and Y. Wang, *Mathematics of Control, Signals, and Systems*, 1996

Lyapunov characterizations

$$\dot{x} = f_\sigma(x, w), \qquad x(0) = x_0$$

- A common Lyapunov function
- Multiple Lyapunov functions



The switched system is GAS if

- each mode admits a Lyapunov function V_p which decreases along the solution when that mode is active:
$$D_{f_p} V_p(x, w) \leq -\lambda V_p(x),$$
- and their values at switches are decreasing:
$$V_{\sigma(t_k)}(x(t_k)) \leq V_{\sigma(t_l)}(x(t_l))$$
for all switches $t_k > t_l$.

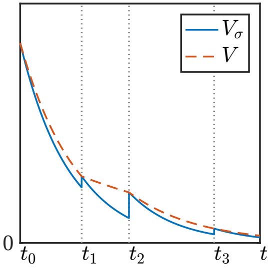
[PD91] P. Peleties and R. DeCarlo, in *1991 American Control Conference*, 1991

[Bra98] M. S. Branicky, *IEEE Transactions on Automatic Control*, 1998

Lyapunov characterizations

$$\dot{x} = f_\sigma(x, w), \qquad x(0) = x_0$$

- A common Lyapunov function
- Multiple Lyapunov functions
- Dwell-time [Mor96], Average dwell-time (ADT) [HM99]



The switched system is GAS if

- each mode admits a Lyapunov function V_p which decreases along the solution when that mode is active:

$$D_{f_p} V_p(x, w) \leq -\lambda V_p(x),$$

- their values after each switch is bounded in ratio: $\exists\, \mu \geq 1$ s.t.

$$V_p(x) \leq \mu V_q(x),$$

- there is an ADT $\tau_a > \ln(\mu)/\lambda$ with an integer $N_0 \geq 1$:

$$N_\sigma(t, \tau) \leq N_0 + (t - \tau)/\tau_a.$$

[Mor96] A. S. Morse, *IEEE Transactions on Automatic Control*, 1996

[HM99] J. P. Hespanha and A. S. Morse, in *38th IEEE Conference on Decision and Control*, 1999

Lyapunov characterizations

$$\dot{x} = f_\sigma(x, w), \qquad x(0) = x_0$$

- A common Lyapunov function
- Multiple Lyapunov functions
- Dwell-time, ADT
- Further results under slow switching:
  - ISS with dwell-time [XWL01]
  - ISS and integral-ISS with ADT [VCL07]
  - ISS and IOSS with ADT [ML12]

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[XWL01] W. Xie, C. Wen, and Z. Li, *IEEE Transactions on Automatic Control*, 2001
[VCL07] L. Vu, D. Chatterjee, and D. Liberzon, *Automatica*, 2007
[ML12] M. A. Müller and D. Liberzon, *Automatica*, 2012

Interconnected switched systems

- An interconnection of switched systems with state $x = (x_1, x_2)$ and external disturbance w

$$\dot{x}_1 = f_{1,\sigma_1}(x_1, x_2, w),$$
$$\dot{x}_2 = f_{2,\sigma_2}(x_1, x_2, w).$$

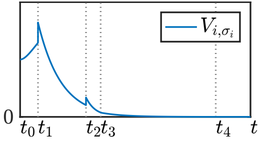
- Each x_i-subsystem regards x_j as internal disturbance
- The switchings σ_1, σ_2 are independent
- Each x_i-subsystem has stabilizing modes in $\mathcal{P}_{s,i}$ and destabilizing ones in $\mathcal{P}_{u,i}$
- Objective: establish ISpS of the interconnection using
  - Generalized ISpS-Lyapunov functions
  - Average dwell-times (ADT)
  - Time-ratios
  - A small-gain condition

Assumptions

- (Generalized ISpS-Lyapunov) For each x_i-subsystem
  - Each $p_s \in \mathcal{P}_{s,i}$ admits an ISpS-Lya function V_{i,p_s} that decreases when active and

$$V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)),\ \chi_i^w(|w|),\ \delta_i\};$$

  - Each $p_u \in \mathcal{P}_{u,i}$ admits a function V_{i,p_u} that may increase when active;
  - $V_{i,p}(x_i) \leq \mu_i V_{i,q}(x_i)$ for all $p, q \in \mathcal{P}_i$.
- (ADT) There is a large enough ADT $\tau_{a,i}$.
- (Time-ratio) There is a small enough time-ratio $\rho_i \in [0, 1)$.



Lyapunov-based small-gain theorem

- (Generalized ISpS-Lyapunov) For each x_i-subsystem
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    $$V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)),\ \chi_i^w(|w|),\ \delta_i\};$$
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- (ADT) There is a large enough ADT $\tau_{a,i}$.
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Theorem (Small-Gain)

The interconnection is ISpS provided that the *small-gain* condition

$$\chi_1^* \circ \chi_2^* < \mathrm{Id}$$

holds with $\chi_i^* := e^{\Theta_i}\chi_i$.

Proof: Hybrid ISpS-Lyapunov function

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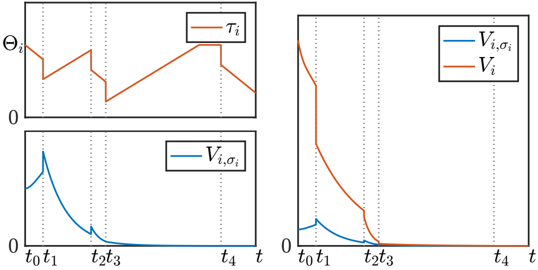
- (ADT) There is a large enough ADT $\tau_{a,i}$.
- (Time-ratio) There is a small enough time-ratio $\rho_i \in [0, 1)$.
- Auxiliary timer $\tau_i \in [0, \Theta_i]$
- Hybrid ISpS-Lya for subsystem $V_i := V_{i,\sigma_i} e^{\tau_i}$ with $\chi_i^* := e^{\Theta_i} \chi_i$



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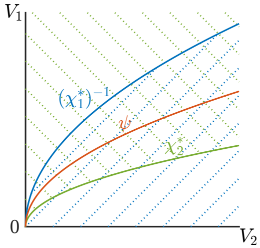
- Small-gain $\chi_1^* \circ \chi_2^* < \mathrm{Id}$
- Hybrid ISpS-Lya function $V := \max\{\psi(V_1),\ V_2\}$



[JMW96] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, *Automatica*, 1996

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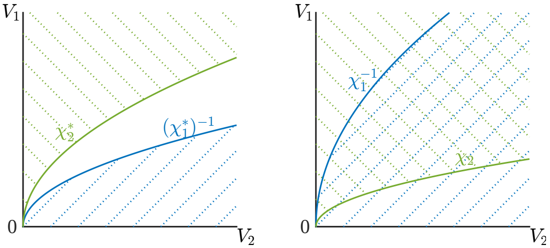

Stabilization via a small-gain approach

- An interconnection of switched control-affine systems with state $x = (x_1, x_2)$, external disturbance w and controls u_1, u_2

$$\dot{x}_1 = f^0_{1,\sigma_1}(x, w) + G_{1,\sigma_1}(x, w)\, u_1,$$
$$\dot{x}_2 = f^0_{2,\sigma_2}(x, w) + G_{2,\sigma_2}(x, w)\, u_2.$$

- (Generalized ISS-Lyapunov) For each x_i-subsystem in open-loop
  - Each $p_s \in \mathcal{P}_{s,i}$ admits an ISS-Lya function V_{i,p_s} that decreases when active and

$$V_{i,p_s}(x_i) \geq \max\{\chi^0_i(V_{j,p_j}(x_j)),\ \chi^w_i(|w|)\};$$

Proposition (Gain assignment)

For each $\chi_i \in \mathcal{K}_\infty$ and $\delta_i > 0$, there is a feedback control $u_i = -\nu_{i,\sigma_i}(|x_i|)\nabla V_{i,\sigma_i}$ s.t. in closed-loop, each V_{i,p_s} decreases when active and

$$V_{i,p_s}(x_i) \geq \max\{\chi_i(V_{j,p_j}(x_j)),\ \chi^w_i(|w|),\ \delta_i\}.$$

[JM97] Z.-P. Jiang and I. M. Y. Mareels, *IEEE Transactions on Automatic Control*, 1997

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$$V_{i,p_s}(x_i) \geq \max\{\chi^0_i(V_{j,p_j}(x_j)),\ \chi^w_i(|w|)\};$$

Theorem

For each $\varepsilon > 0$, there are feedback controls u_1, u_2 s.t. the interconnection in closed-loop is ISpS with the constant ε, i.e., there are $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ s.t.

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|w\|) + \varepsilon \qquad \forall\, t \geq 0.$$

Summary

- Interconnected switched systems
- A Lyapunov-based small-gain theorem for ISpS
  - Hybrid ISpS-Lyapunov functions and a small-gain condition
  - Increase in internal gains due to switching and destabilizing modes
- Stabilization via a small-gain approach
  - Gain assignment using feedback control