Topological Entropy of Switched Nonlinear Systems

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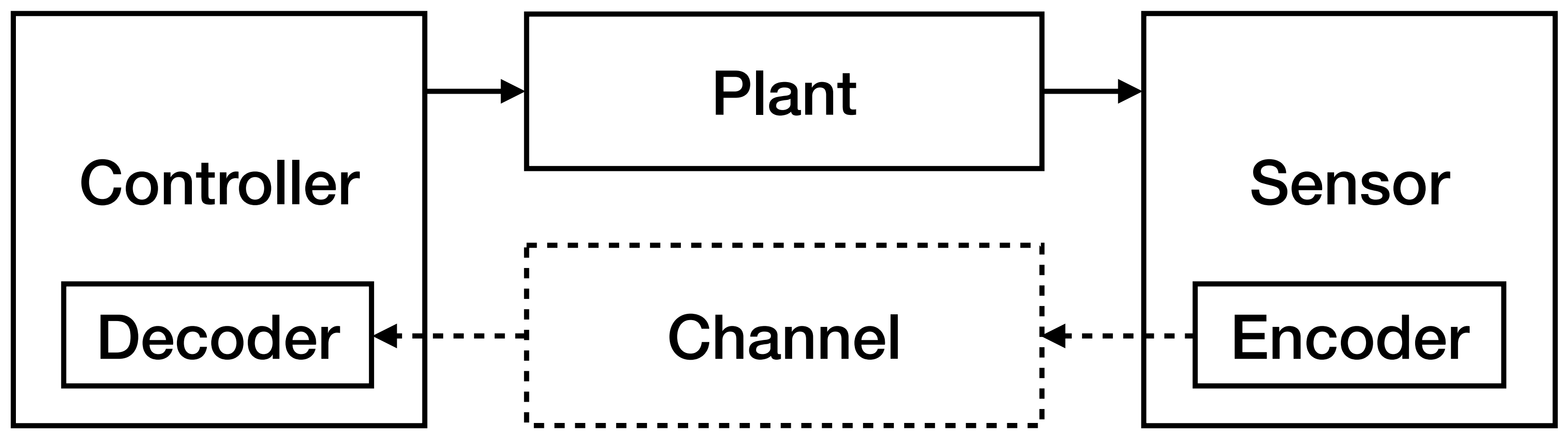
Motivation: How Much Data Rate Is Needed for Control?

Control over digital communication:

- Sensor collects information about state/output
- Information is encoded for digital transmission
- Transmission is decoded to generate control input for tasks such as stabilization, ensuring set invariance, etc.



How much data rate is needed?

- Described by topological entropy and variants
- Complexity: exponential growth rate of # of distinguishable trajectories

Entropy notions in systems and control:

- Topological entropy [Adler-Konheim-McAndrew'65; Bowen'71; Dinaburg'70]
- Nonautonomous systems [Kolyada-Snoha'96; Kawan-Latushkin'16]
- Switched linear systems [**Y**-Schmidt-Liberzon-Hespanha'20; Berger-Junger'20]
- Control entropy [Nair-Evans-Mareels-Moran'04; Colonious-Kawan'09; Colonius'12]
- Estimation entropy [Savkin'06; Matveev-Pogromsky'16; Liberzon-Mitra'18]

- Minimal # of trajectories needed to approximate all trajectories with increasing precision

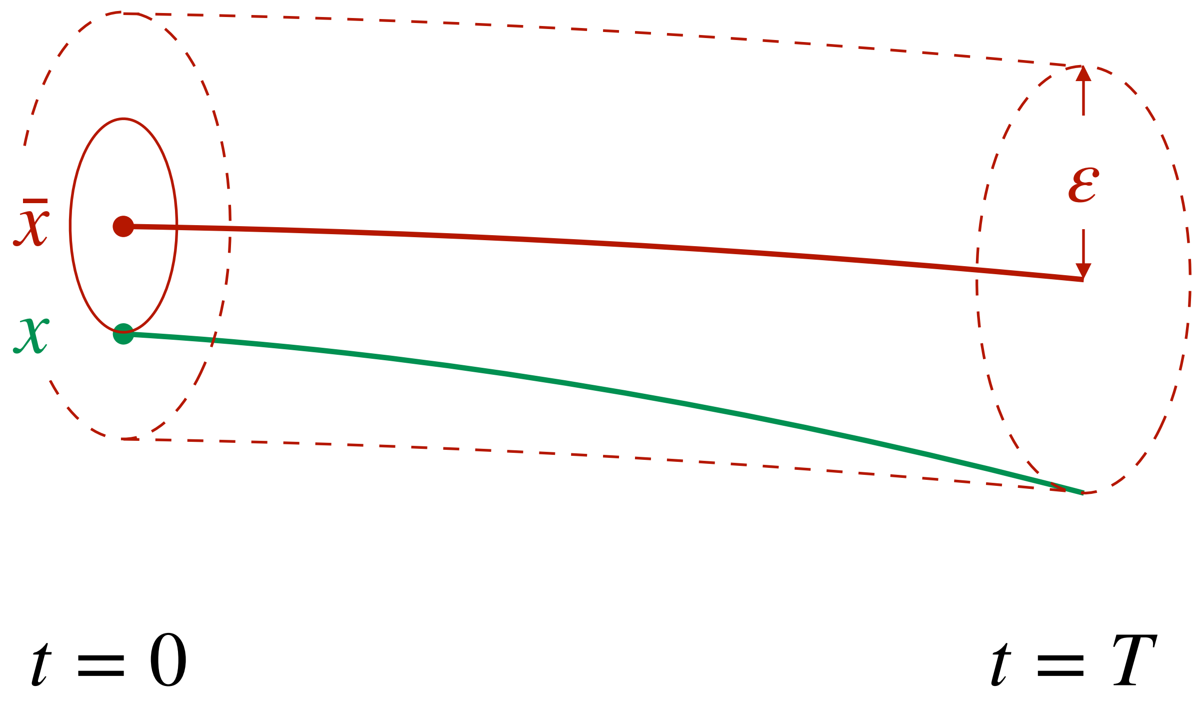
Entropy Definition

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, x(0) \in K$$

- $K \subset \mathbb{R}^n$: known initial set, compact with nonempty interior
- $\xi(t, x)$: solution at time t with initial state x

Entropy definition:

- Pick norm $\|\cdot\|$, time horizon $T \geq 0$ and resolution $\varepsilon > 0$ (eventually $T \to \infty$ and $\varepsilon \searrow 0$)
- A set E of initial states is (T, ε)-spanning if $\forall x \in K\, \exists \bar{x} \in E : \max_{t \in [0,T]} \|\xi(t, x) - \xi(t, \bar{x})\| < \varepsilon$



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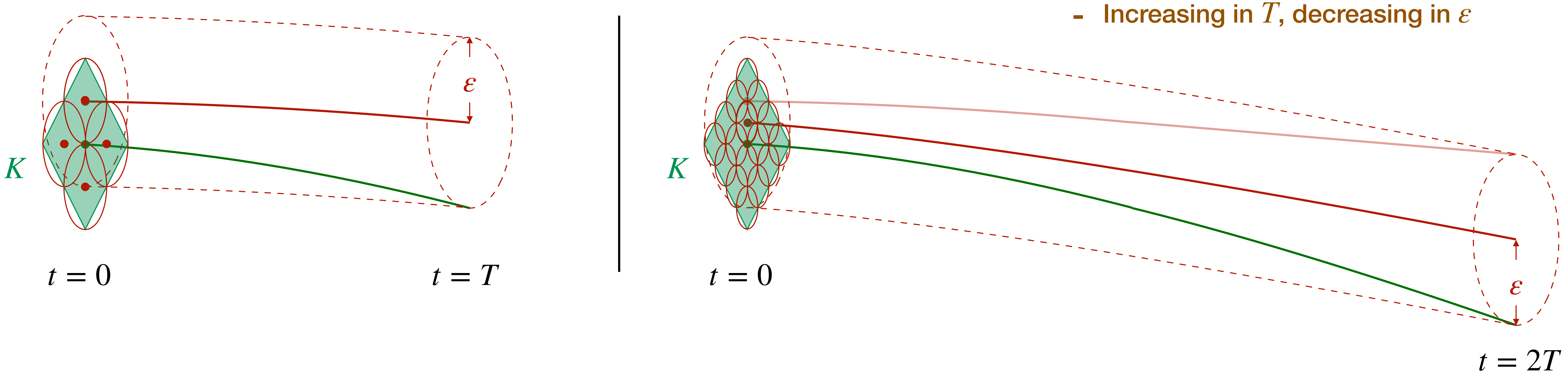
- $S(\varepsilon, T, K)$: minimal cardinality of a (T, ε)-spanning set
  - Minimal # of trajectories needed to approximate all trajectories from K with error $< \varepsilon$ over $[0,T]$
  - Increasing in T, decreasing in ε



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- $S(\varepsilon, T, K)$: minimal cardinality of a (T, ε)-spanning set
- Topological entropy: exponential growth rate of $S(\varepsilon, T, K)$

- Minimal # of trajectories needed to approximate all trajectories from K with error $< \varepsilon$ over $[0,T]$
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$$h = \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(\varepsilon, T, K)$$

- $h \geq 0$
- Entropy bounds on the slides actually mean the maximum of them and zero

Intuition:

- E is a set of quantization points (with error $< \varepsilon$)
- $\log S(\varepsilon, T, K)$ corresponds to the minimal number of bits needed to specify one quantization point
- h corresponds to the minimal bit rate for quantization

Entropy of Linear Time-Invariant Systems

Topological entropy

$$h = \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(\varepsilon, T, K)$$

Linear time-invariant (LTI) system $\dot{x} = Ax$:

Topological entropy $h = \sum_{\lambda \in spec(A)} \max\{Re(\lambda), 0\}$ = Minimal data rate for stabilization

- Entropy formula: [Bowen'71; Colonius-Kawan'09]
- Minimal data rate for stabilization: [Hespanha-Ortega-Vasudevan'02; Nair-Evans'03; Tatikonda-Mitter'04]

Switched Systems

$$\dot{x} = f_\sigma(x) \qquad x \in \mathbb{R}^n, x(0) \in K$$

- Modes $\{f_1(x), \ldots, f_P(x)\}$; $\sigma : \mathbb{R}_{\geq 0} \to \{1, \ldots, P\}$: piecewise constant switching signal
- Solution: $\xi_\sigma(t, x) = \cdots \underline{\xi_{p_2}(t_2 - t_1, \xi_{p_1}(t_1, x))} \cdots$

Bound for distance between solutions:

- Matrix measure: $\mu(A) := \lim_{t \searrow 0} \frac{\|I + tA\| - 1}{t}$
  - Right-hand derivative of $\|e^{At}\|$ at $t = 0$
  - $Re(\lambda) \leq \mu(A) \leq \|A\|$; can have $\mu(A) < 0$
- For LTV system $\dot{x} = A(t)\,x$, it is well-known that $\|x(t)\| \leq e^{\int_{t_0}^{t} \mu(A(s))\,\mathrm{d}s} \|x(0)\|$
  - Integral of the measure of system matrix
- **Proposition 2.5.**

$$\|\xi_\sigma(t, x) - \xi_\sigma(t, \bar{x})\| \leq e^{\bar{\eta}(t)} \|\bar{x} - x\| \qquad \text{with } \bar{\eta}(t) := \max_{v \in co(K)} \int_0^t \mu(J_x f_{\sigma(s)}(\xi_\sigma(s, v)))\,\mathrm{d}s$$

- Integral of the measure of Jacobian along trajectory

Sketch of proof: Variational method

- Write distance as an integral of Jacobian $J_x \xi_\sigma(t, v)$ over the line segment
- Write $J_x \xi_\sigma(t, x)$ as the state of an LTV, apply the above bound

Similar lower bound for volume of reachable set $\xi_\sigma(t, K)$

Entropy of Switched Systems

$$\dot{x} = f_\sigma(x) \qquad x \in \mathbb{R}^n, x(0) \in K$$

- Topological entropy h is defined for a fixed switching signal σ, similarly as before

Useful quantities about switching:

- Active time of mode p: $\tau_p(t) = \int_0^t \mathbf{1}_p(\sigma(s))\,\mathrm{d}s$ with $\mathbf{1}_p(\sigma(s)) = 1$ if $\sigma(s) = p$ and 0 if not
- Active rate $\rho_p(t) = \tau_p(t)/t$; asymptotic active rate $\hat{\rho}_p = \limsup_{t\to\infty} \rho_p(t)$
  - $\sum_p \rho_p(t) \equiv 1$; can have $\sum_p \hat{\rho}_p > 1$

Entropy of switched linear system $\dot{x} = A_\sigma x$ [**Y**-Schmidt-Liberzon-Hespanha'20; **Y**-H-L'19]:

- General upper/lower bound:

$$\limsup_{t\to\infty} \sum_p \mathrm{tr}(A_p)\rho_p(t) \le h \le \limsup_{t\to\infty} \sum_p n\mu(A_p)\rho_p(t)$$

  - Asymptotic average of the measure/trace of system matrices, weighted by active rates $\rho_p(t)$
- An exact formula for commuting matrices (i.e., $A_pA_q = A_qA_p$)
- Connections with stability: e.g.,

$$h(A_\sigma + \delta I) = 0 \text{ for some } \delta > 0 \implies \text{stable switched system}$$

Entropy of Switched Nonlinear Systems

Theorem 3.1. General upper bound:

$$h \leq \limsup_{t\to\infty} \sum_p n\hat{\mu}_p \rho_p(t) \qquad \text{with } \hat{\mu}_p = \limsup_{s\to\infty} \max_{v\in co(K)} \mu(J_x f_p(\xi_\sigma(s,v)))$$

Feature:

- Asymptotic average of $n\hat{\mu}_p$, weighted by active rates $\rho_p(t)$
- $\hat{\mu}_p$: supremum of the measure of Jacobian matrix over the ω-limit set

Sketch of proof:

- **Lemma 2.3.** Constructing standard spanning sets to show:

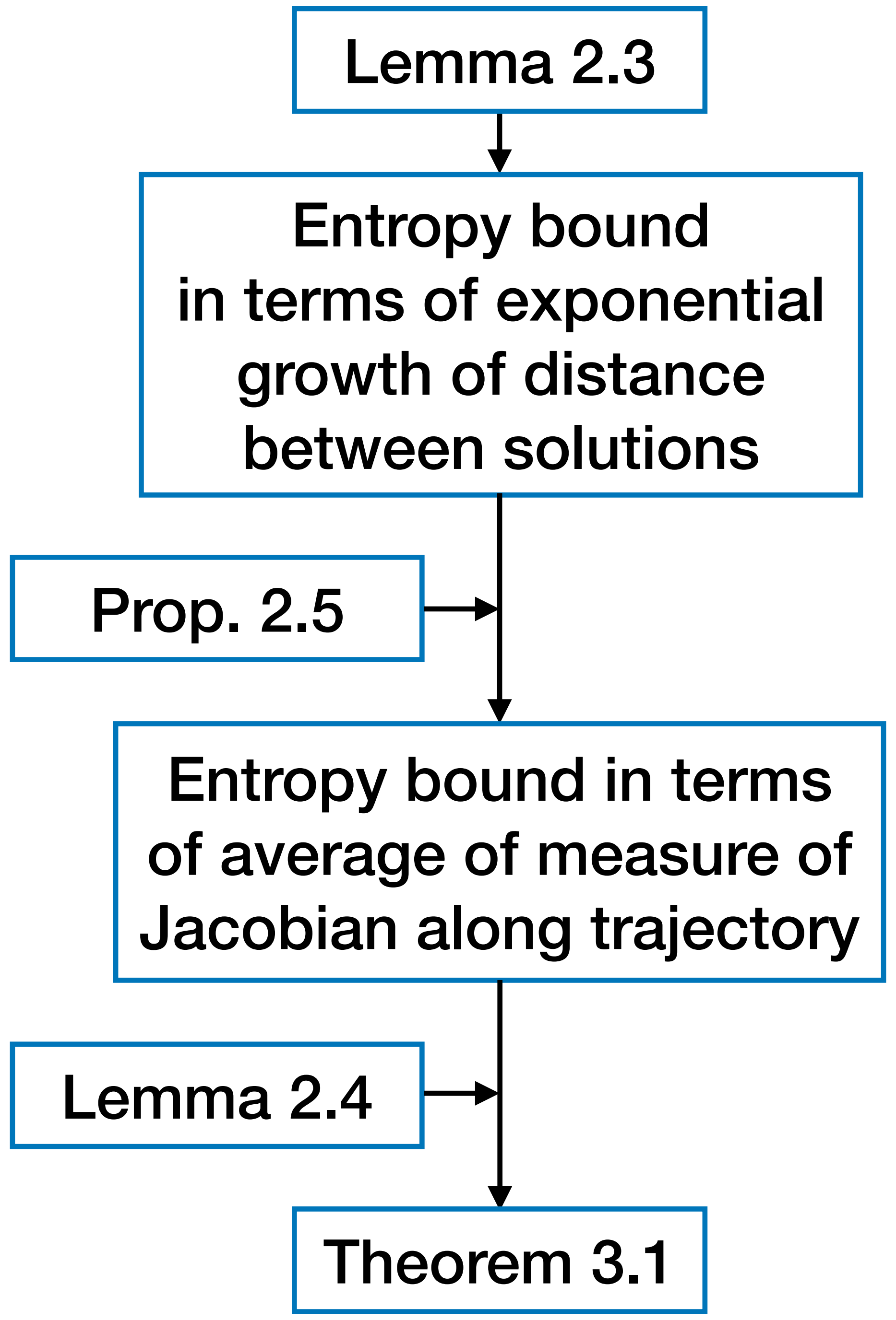
  If $\|\xi_\sigma(t,x) - \xi_\sigma(t,\bar{x})\| \leq e^{\bar{\eta}(t)}\|\bar{x} - x\|$, then $h \leq \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \max_{t\in[0,T]} n\bar{\eta}(t)$
- **Proposition 2.5.** Bound for distance between solutions:

  $\|\xi_\sigma(t,x) - \xi_\sigma(t,\bar{x})\| \leq e^{\bar{\eta}(t)}\|\bar{x} - x\|$ with $\bar{\eta}(t) := \max_{v\in co(K)} \int_0^t \mu(J_x f_{\sigma(s)}(\xi_\sigma(s,v)))\,\mathrm{d}s$
- **Lemma 2.4.** Separating coefficients of system dynamics and switching:

  For integrable functions $\{a_1(s), \ldots, a_P(s)\}$,

  $$\limsup_{T\to\infty} \frac{1}{T} \max_{t\in[0,T]} \int_0^t a_{\sigma(s)}(s)\,\mathrm{d}s \leq \limsup_{t\to\infty} \sum_p \left(\limsup_{s\to\infty} a_p(s)\right) \rho_p(t)$$

  - So that $\hat{\mu}_p$ only depending on ω-limit set



Entropy of Switched Nonlinear Systems

Theorem 3.1. General upper bound:

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Theorem 3.1. General lower bound:

$$h \geq \limsup_{t\to\infty} \sum_p \check{\chi}_p \rho_p(t) \qquad \text{with } \check{\chi}_p = \liminf_{s\to\infty} \min_{v\in K} \mathrm{tr}(J_x f_p(\xi_\sigma(s, v)))$$

Feature:

- Asymptotic average of $\check{\chi}_p$ weighted by active rates
- $\check{\chi}_p$: infimum of the trace of Jacobian over the ω-limit set

Proof: bound for volume of reachable set $\xi_\sigma(t, K)$

Entropy of Switched Nonlinear Systems

Theorem 3.1. General upper bound:

$$h \leq \limsup_{t\to\infty} \sum_p n\hat{\mu}_p \rho_p(t) \qquad \text{with } \hat{\mu}_p = \limsup_{s\to\infty} \max_{v\in co(K)} \mu(J_x f_p(\xi_\sigma(s,v)))$$

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Corollary 3.2. More conservative upper bounds that require less information about switching:

- Depending on asymptotic active rates $\hat{\rho}_p$ instead of active rates $\rho_p(t)$

$$h \leq \sum_p n\hat{\mu}_p\hat{\rho}_p, \qquad h \leq \max_p n\hat{\mu}_p$$

- Does not involve active rates at all

Theorem 4.1, Corollary 4.2. Tighter bounds for entropy of switched diagonal systems $\dot{x}_i = f^i_\sigma(x_i)$

Numerical Example

Switched Lotka—Volterra ecosystem

$$\dot{x}_i = f_\sigma(x) = \left(r_\sigma^i + \sum_j a_\sigma^{ij} x_j\right) x_i, \qquad x \in \mathbb{R}^n_{\geq 0}$$

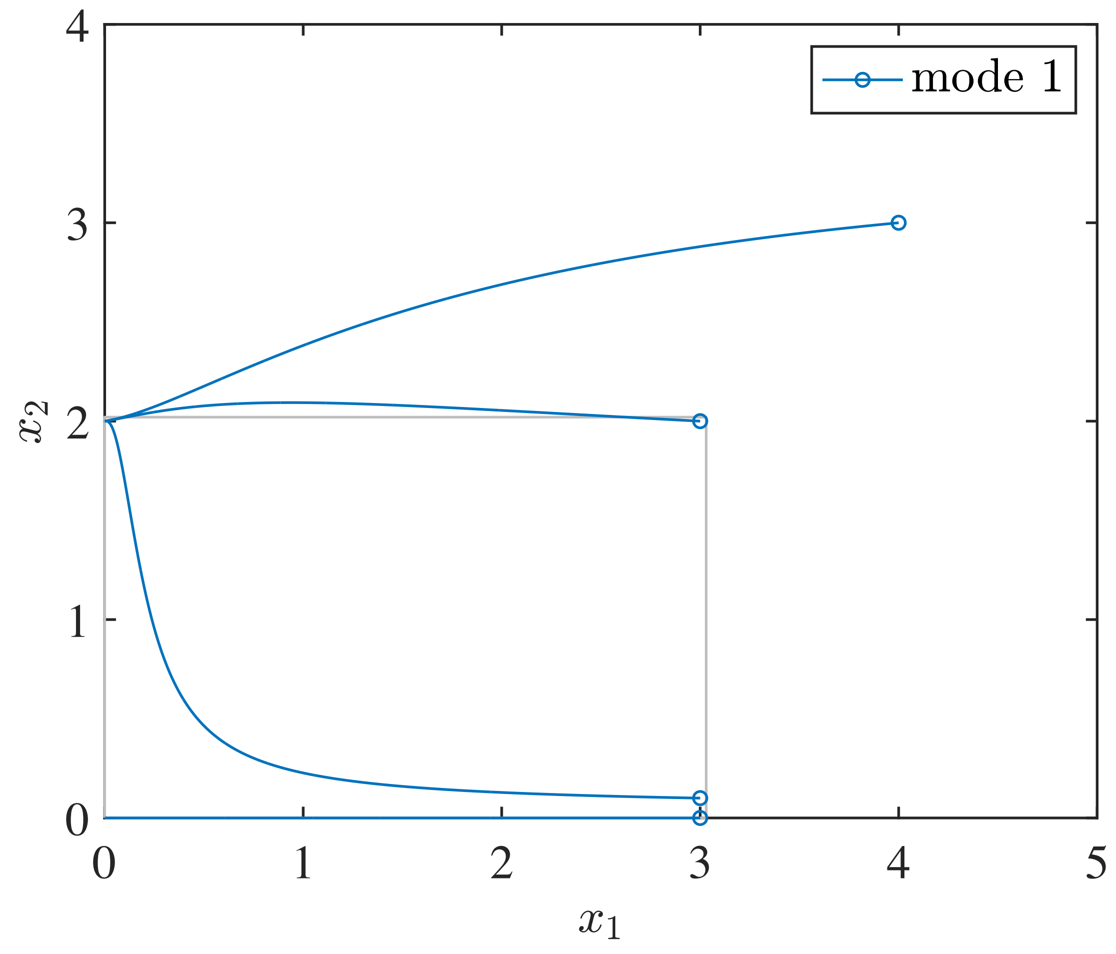
- x_i: population density of the i-th species
- r_p^i: intrinsic growth rate of the i-th population
- $a_p^{ii} < 0$: self-interaction term due to limited resource
- a_p^{ij}: influence of the j-th population on the i-th one
- Switching may be due to seasonal changes or other environmental factors

Numerical Example

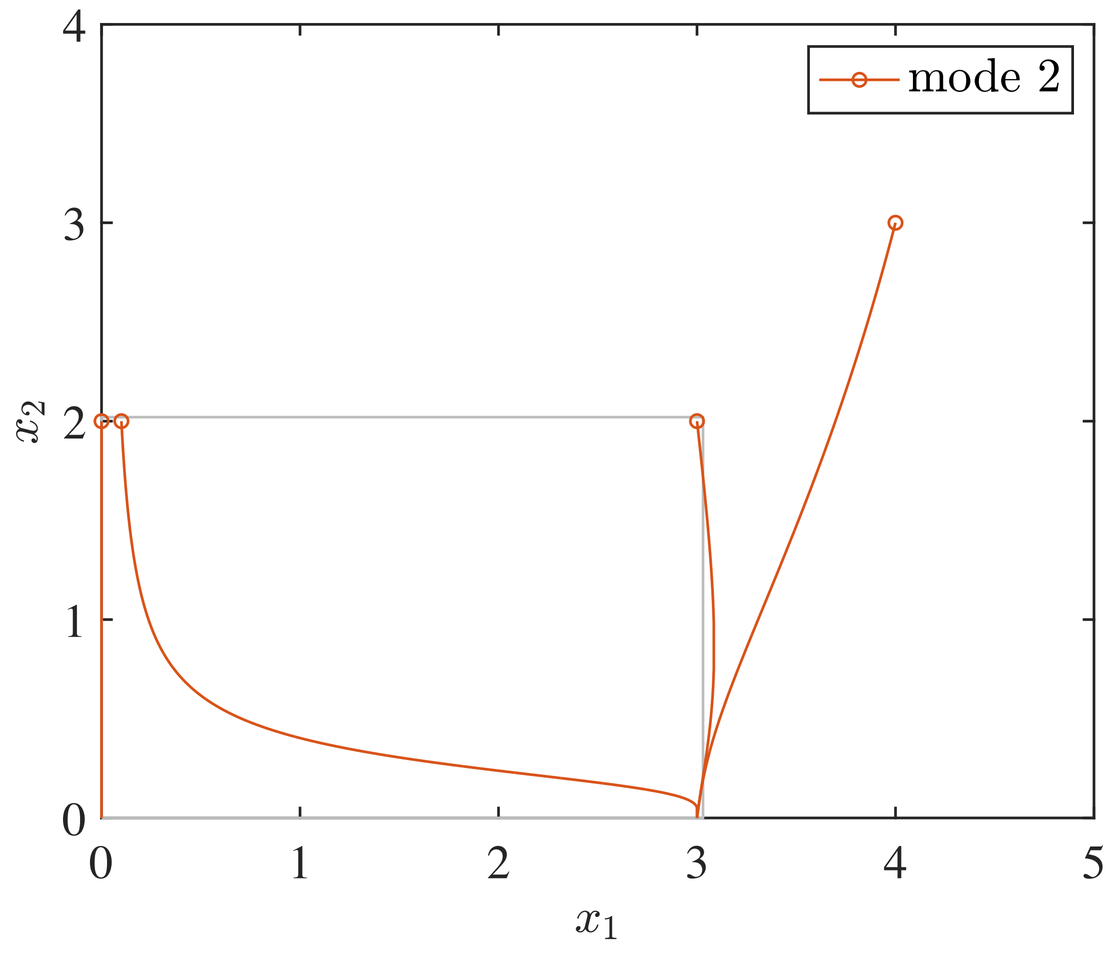
Switched Lotka—Volterra ecosystem with two species

Mode 1:
$$\dot{x}_1 = (-1 - x_1 + 0.1x_2)\,x_1$$
$$\dot{x}_2 = (2 + 0.1x_1 - x_2)\,x_2$$

Mode 2:
$$\dot{x}_1 = (3 - x_1 + 0.1x_2)\,x_1$$
$$\dot{x}_2 = (-1 + 0.1x_1 - x_2)\,x_2$$



Attractor $(0{,}2)$; Saddle point $(0{,}0)$ with stable manifold $\mathbb{R}_{\geq 0} \times \{0\}$



Attractor $(3{,}0)$; Saddle point $(0{,}0)$ with stable manifold $\{0\} \times \mathbb{R}_{\geq 0}$

Convergence of switched system [Aleksandrov-Chen-Platonov-Zhang'11]:

- ω-limit set $\subset \Omega := [0{,}3.04] \times [0{,}2.03]$ (gray rectangle) - Ω is not positively invariant

Numerical Example

Switched Lotka—Volterra ecosystem with two species

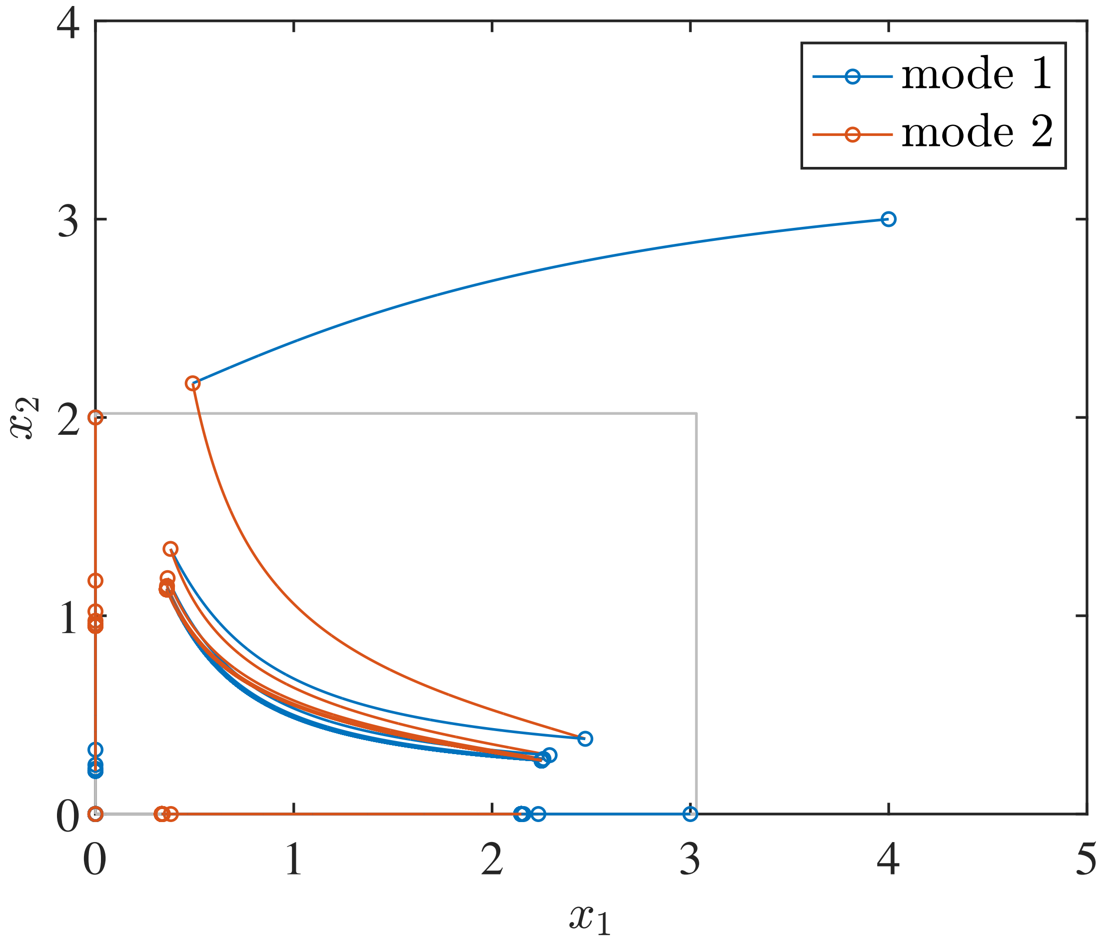
Mode 1: $\dot{x}_1 = (-1 - x_1 + 0.1x_2)\,x_1$, $\dot{x}_2 = (2 + 0.1x_1 - x_2)\,x_2$

Mode 2: $\dot{x}_1 = (3 - x_1 + 0.1x_2)\,x_1$, $\dot{x}_2 = (-1 + 0.1x_1 - x_2)\,x_2$

Switching signals:

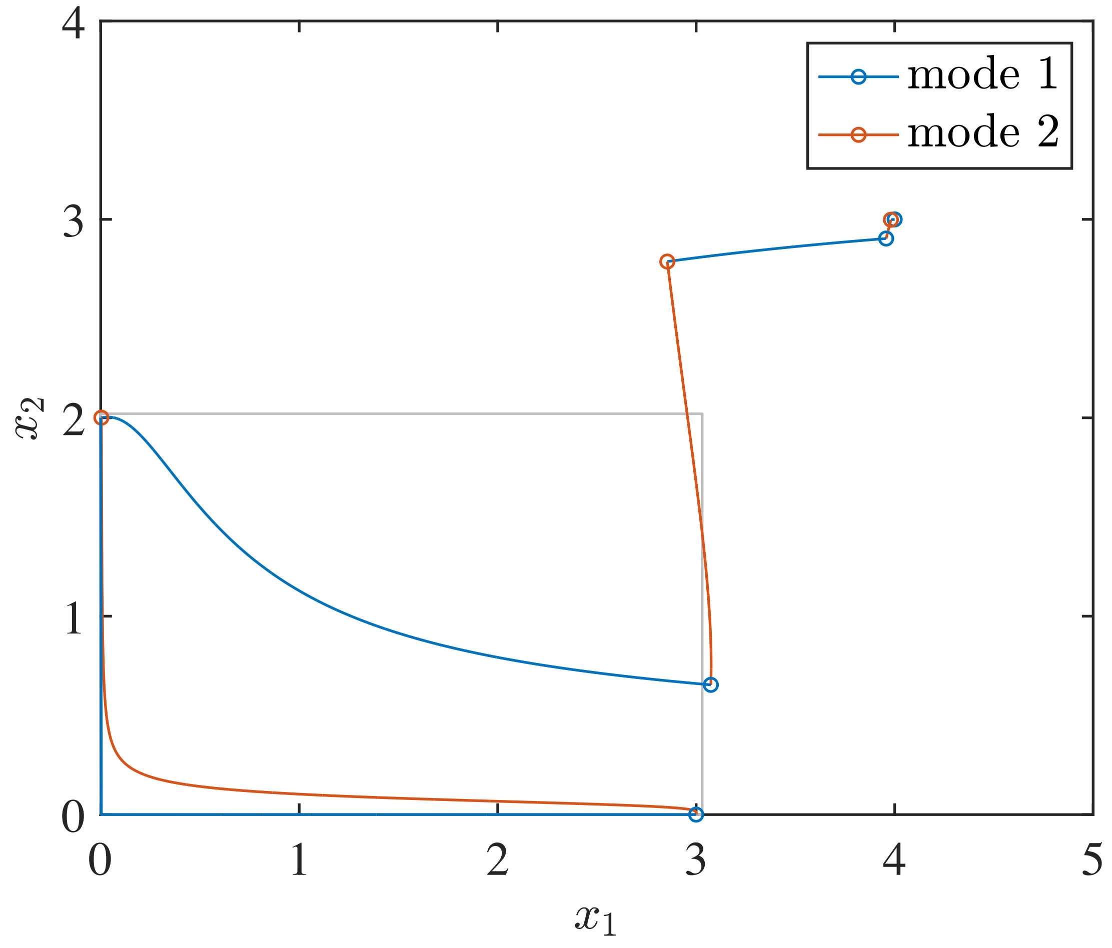
σ_1: switching periodically
switches: $\{1000, 2000, \ldots, 1000k, \ldots\}$
asymptotic active rates: $\hat{\rho}_1 = \hat{\rho}_2 = 0.5$

σ_2: switching when $\rho_{\tau(t)}(t)$ reaches 0.9
switches: $\{1, 10, 90, \ldots, 10 \times 9^{k-2}, \ldots\}$
asymptotic active rates: $\hat{\rho}_1 = \hat{\rho}_2 = 0.9$



	$(\hat{\rho}_1, \hat{\rho}_2)$	Th. 3.1	Cor. 3.2	
σ_1	$(0.5, 0.5)$	5.52	5.52	6.42
σ_2	$(0.9, 0.9)$	6.24	9.94	6.42

- Th. 3.1: $\limsup_{t\to\infty} \sum_p n\hat{\mu}_p \rho_p(t)$
- Cor. 3.2: (1) $\sum_p n\hat{\mu}_p \hat{\rho}_p$; (2) $\max_p n\hat{\mu}_p$
- ($\hat{\mu}_p$ computed over Ω)



- Switching prevents extinction
- Entropy describes data-rate requirements for monitoring

Conclusion

Summary:

- General upper/lower bounds for topological entropy of switched nonlinear systems
- More conservative upper bounds that require less information about switching
- Tighter bounds for switched diagonal systems
- Feature: most bounds only depend on Jacobian over ω-limit set
- Numerical example of a switched Lotka—Volterra ecosystem

Future research:

- Topological entropy of nonlinear time-varying systems
- Topological entropy of switched commuting systems
- Connections between topological entropy and stability

Thank you!