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Stabilizing a switched linear system with disturbance by sampled-data quantized feedback

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Abstract— **We study the problem of stabilizing a switched linear system with disturbance using sampled and quantized measurements of its state. The switching is assumed to be slow in the sense of combined dwell-time and average dwell-time, while the active mode is unknown except at sampling times. Each mode of the switched linear system is assumed to be stabilizable, and the magnitude of the disturbance is constrained by a known bound. A communication and control strategy is designed to guarantee bounded-input-bounded-state (BIBS) stability of the switched linear system and an exponential convergence rate with respect to the initial state, providing the data rate satisfies certain lower bounds. Such lower bounds are established by expanding the over-approximation bounds of reachable sets over sampling intervals derived in a previous paper to accommodate effects of the disturbance.**

I. Introduction

Feedback control problems with limited information have been an active research area for years, as surveyed by Nair et al. [1]. Information flow in a feedback loop has been an important factor in many application-related scenarios, not only because of bandwidth constraints, but for cost concerns, physical restrictions, and security considerations as well. Besides the aforementioned practical motivations, the question of how much information is required to achieve a certain control objective is quite fundamental and intriguing from the theoretical point of view. In the study of feedback control problems, it is common to characterize the limitation on information flow as a finite data transmission rate achieved by using sampled and quantized measurements to generate the control input (see, e.g., [2], [3] and [4, Ch. 5]), which is the modeling framework adopted in this paper.

We are interested in feedback control problems using sampled and quantized measurements in the presence of external disturbances. In this context, the work by Hespanha et al. [2] and Tatikonda and Mitter [3] assumed known bounds on the magnitudes of external disturbances and addressed bounded-input-bounded-state (BIBS) stability [5], while the work by Liberzon and Nešić [6] and Sharon and Liberzon [7] avoided such assumptions by switching repeatedly between "zooming-out" and "zooming-in" processes and achieved input-to-state stability (ISS) [8]. See also [9], [10] for related results in a stochastic setting.

The study of switched and hybrid systems has attracted lots of attention in recent years (particularly relevant works include [4], [11], [12] and many references therein). In the research on stability and stabilization of switched systems, it is common to impose certain slow-switching conditions, especially in the sense of dwell-time [13] and average dwell-time [14], which play a crucial role in our analysis.

Towards stabilization of switched systems with disturbances, Hespanha and Morse [14] showed that ISS is preserved under the same average dwell-time condition as for the case without disturbance. This result was made explicit in [14] only for the case of switched linear systems, and several papers have established similar results in the context of switched nonlinear systems since then (e.g., [15] for ISS with dwell-time, [16] for ISS with average dwell-time, [17] for input/output-to-state stability with average dwell-time).

Early work on control problems with limited information in the context of switched systems has been devoted to quantized control of Markov jump linear systems [18], [19], [20]. However, the discrete modes in these references were always known to the controller, which would remove most of the difficulties present in our problem formulation. The problem of asymptotically stabilizing a switched linear system using sampled and quantized state feedback was studied in [21], which serves as the basis of the present work. In [21], the controller was assumed to have a partial knowledge of the switching; namely, the switching signal was subject to a fairly mild "slow-switching" condition described by the combination of a dwell-time and an average dwell-time, while the active mode of the switched system was unknown except at sampling times. Providing the data rate was large enough (but finite), a communication and control strategy was developed based on the technique of propagating over-approximations of reachable sets. A related result for output feedback stabilization was presented in [22].

We extend the result in [21] to the scenario where an external disturbance is present. The disturbance is unknown but an upper bound on its magnitude is known to the controller. We design and verify a communication and control strategy which guarantees BIBS stability of the switched linear system and an exponential convergence rate with respect to the initial state, assuming the data rate satisfies certain lower bounds. While such lower bounds are established based on the propagation of over-approximations of reachable sets during sampling intervals along the lines of [21], the bounds on reachable sets are enlarged accordingly to accommodate effects of the disturbance. Moreover, the lower bound on the data rate guaranteeing BIBS stability is formulated explicitly, together with a remark discussing its relation to the lower bound guaranteeing exponential convergence.

This paper is structured as follows. In Section II, we intro-

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duce the definitions and basic assumptions of the switched linear system with disturbance and the information structure. Our main result is presented in Section III. The communication and control strategy is described in Section IV under the assumption that appropriate bounds on reachable sets are available. Such bounds are constructed in Section V. In Section VI, we sketch the stability analysis, with several major steps summarized as technical lemmas. Section VII summarizes the paper and introduces a future research topic.

II. Problem Formulation

A. System description

In this paper, we are interested in stabilizing a switched linear control system with disturbance

$$\dot{x} = A_\sigma x + B_\sigma u + D_\sigma d, \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the (control) input, $d \in \mathbb{R}^{n_d}$ is the external disturbance, $\{(A_p, B_p, D_p) : p \in \mathcal{P}\}$ is a collection of matrix triples with suitable dimensions defining the subsystems (modes), $\mathcal{P}$ is a finite *index set*, and $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ is a right-continuous, piecewise constant *switching signal* which specifies the index $\sigma(t)$ of the active mode at time t. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation (1) away from the discontinuities of σ (in particular, there are no state jumps). The switching signal σ is fixed but unknown to the controller a priori. Discontinuities of σ are called *switching times*, or simply *switches*. The number of switches on a time interval $(s, t]$ is denoted by $N_\sigma(t, s)$.

Our first basic assumption is that σ satisfies a "slow-switching" condition, which is characterized by the following combined dwell-time and average dwell-time conditions.

Assumption 1 (Slow switching)**.** The switching signal σ satisfies that

1) there exists a *dwell-time* τ_d such that $N_\sigma(t, s) \leq 1$ for all $s \in \mathbb{R}_{\geq 0}$ and all $t \in (s, s + \tau_d]$;
2) there exists an *average dwell-time* $\tau_a > \tau_d$ and an integer $N_0 \geq 1$ such that

$$N_\sigma(t, s) \leq N_0 + \frac{t - s}{\tau_a} \qquad \forall\, t > s \geq 0. \tag{2}$$

The notions of dwell-time and average dwell-time were introduced by Morse [13] and Hespanha and Morse [14], respectively, and are quite standard in the context of switched systems. In Assumption 1, item 1) may be represented in the form of (2) with $\tau_a = \tau_d$ and $N_0 = 1$; on the other hand, item 2) would be implied by item 1) in the absence of the constraint $\tau_a > \tau_d$. Switching signals satifying Assumption 1 were called "hybrid dwell-time" signals in [23].

Our second basic assumption is that all individual modes are stabilizable.

Assumption 2 (Stabilizability)**.** For each $p \in \mathcal{P}$, there exists a state feedback gain matrix K_p such that $F_p := A_p + B_p K_p$ is Hurwitz.

In the subsequent analysis, it is assumed that such a collection of matrices $\{K_p : p \in \mathcal{P}\}$ has been selected and fixed. In general, even if there is no disturbance, and all individual modes are stabilized via feedback (or stable without feedback), stability of the switched system is not guaranteed (see, e.g., [4, Part II]).

Throughout this work, $\|\cdot\|$ is used to denote the ∞-norm. For a function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, let $f(t^-) := \lim_{s \nearrow t} f(s)$ and $\|f\|_I$ be its supremum ∞-norm on an interval I, that is,

$$\|f\|_I := \sup_{s \in I} \|f(s)\|.$$

Our third basic assumption is that the magnitude of the external disturbance d is bounded by a known value.

Assumption 3 (Disturbance)**.** The disturbance d satisfies that

$$\|d(t)\|_{[0,\infty)} \leq \delta \tag{3}$$

for a *disturbance bound* $\delta \in \mathbb{R}_{\geq 0}$ *known* to both the encoder and decoder.

B. Information structure

The feedback control consists of a sensor and a controller. The sensor records and transmits two sequences of data: the indices of the active modes $\sigma(t_k)$ and the quantized measurements (samples) of the state $x(t_k)$ at *sampling times* $t_k = k\tau_s, k \in \mathbb{Z}_{\geq 0}$, where τ_s is the *sampling period.* Each sample is encoded by an integer i_k from 0 to N^{n_x}, where N is an odd integer (so that the equilibrium at the origin is preserved). The controller generates the input u to the switched linear system (1) based on the decoded data. The information structure of the feedback control is demonstrated in Fig. 1. The communication and control strategy is described in detail in Section IV.

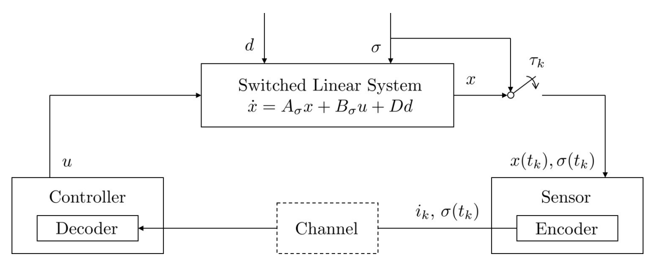

Fig. 1. Information structure

It is assumed that the sampling period is smaller or equal to the dwell-time τ_d in Assumption 1, that is,

$$\tau_s \leq \tau_d, \tag{4}$$

so that at most one switch occurs in each *sampling interval* $(t_k, t_{k+1}]$. (Since the average dwell-time τ_a is larger than τ_d in Assumption 1, switches actually occur less than once per sampling period.) As $\sigma(t_k) \in \mathcal{P}$ and $i_k \in \{0, 1, \ldots, N^{n_x}\}$, the data rate of the transmission between the encoder and decoder is $(\log_2 |N^{n_x} + 1| + \log_2 |\mathcal{P}|)/\tau_s$ bits per time unit, where $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$. Our last basic assumption sets a lower bound on the data rate.

Assumption 4 (Data rate)**.** The sampling period τ_s satisfies

$$\|e^{A_p \tau_s}\| =: \Lambda_p < N \qquad \forall\, p \in \mathcal{P}. \tag{5}$$

Equation (5) is interpreted as a lower bound on the data rate as it requires τ_s to be sufficiently small with respect to N. This bound is the same as the one in the earlier result for the case without disturbance [21, Assumption 3], while similar data rate bounds appeared in [24], [2], [3] for stabilizing non-switched linear systems. (For more discussion on their relation, see [21, Section 2.2] and [7, Section V].)

III. Main result

Our main objective is to establish the following theorem:

Theorem 1. *Let Assumptions 1–4 and the inequality* (4) *hold for the switched linear system* (1)*. If the average dwell-time τ_a is large enough, then there exists a communication and control strategy that yields the following two properties.*

Exponential Convergence: *There exist a constant $\lambda \in \mathbb{R}_{>0}$ and functions $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\|x(t)\| \leq e^{-\lambda t} g(\|x_0\|) + h(\delta) \tag{6}$$

for all initial states $x_0 \in \mathbb{R}^{n_x}$, all disturbance bounds $\delta \in \mathbb{R}_{\geq 0}$ and all $t \in \mathbb{R}_{\geq 0}$.

BIBS Stability: *For each $\epsilon \in \mathbb{R}_{>0}$, there exists a disturbance bound $\delta \in \mathbb{R}_{>0}$ such that if, in addition to the inequality* (3)*, the initial state satisfies $\|x_0\| \leq \delta$, then $\|x(t)\| \leq \epsilon$ for all $t \in \mathbb{R}_{\geq 0}$.*

The lower bounds on the average dwell-time τ_a are given by (33) for exponential convergence, and (42) for BIBS stability. (Their relation is discussed in Remark 2). The *exponential decay rate* λ and the gains g and h are given by (39), (40) and (41), respectively. From the proof, it will be clear that $g(r)$ does not go to 0 as $r \to 0$ and is super-linear as $r \to \infty$. Hence BIBS stability needs to be established separately. Meanwhile, h is super-linear and positive definite.

IV. Communication and control strategy

In this section we present the communication and control strategy under the assumption that appropriate state bounds are available at sampling times. We first explain the "zooming-out" algorithm to capture an arbitrary initial state in Subsection IV-A, and then the "zooming-in" algorithm to measure the state and generate the (control) input in Subsection IV-B. The construction of appropriate state bounds is described in Section V.

A. Capturing the state by "zooming-out"

In the beginning, the sensor possesses no information of the initial state x_0 and is given an arbitrary initial value E_0. Starting from $k = 0$, at each sampling time t_k, it checks if

$$\|x(t_k)\| \leq E_k \tag{7}$$

holds, that is, the state $x(t_k)$ is inside the hypercube $\{v \in \mathbb{R}^{n_x} : \|v\| \leq E_k\}$. If the result is positive, it proceeds to the "zooming-in" stage; otherwise it lets the encoder send $i_k = 0$, the "overflow symbol", to the decoder, calculates a larger value E_{k+1}, and repeats at the next sampling time t_{k+1}. The controller sets the input $u(t) = 0$ on $[t_k, t_{k+1})$ upon receiving $i_k = 0$. The formula to calculate E_{k+1} from E_k and the proof that a finite "capture time" t_{k_0} such that (7) holds with $k = k_0$ exists are presented in Subsection V-C. The decoder knows the initial value E_0 and the recursive formula, and thus calculates every E_{k+1} by itself.

B. Measuring the state by "zooming-in" and generating the control

The state is captured at t_{k_0} for the first time. Hence the inequality (7) holds, that is,

$$\|x(t_k) - x_k^*\| \leq E_k \tag{8}$$

with $k = k_0$ and $x_{k_0}^* = 0$. Starting from $k = k_0$, at each sampling time t_k, both the sensor and the controller know that (8) holds, that is, the state $x(t_k)$ is inside the hypercube $\{v \in \mathbb{R}^{n_x} : \|v - x_k^*\| \leq E_k\}$, and the values of x_k^* and E_k^*. The encoder partitions the hypercube into N^{n_x} equal hypercubic boxes, N per dimension, encodes each box by a unique index from 1 to N^{n_x}, and transmits the index i_k of the hypercubic box containing $x(t_k)$ to the decoder, along with the index $\sigma(t_k)$ of the active subsystem. The decoder shares the same indexing protocol with the encoder, so it is able to reconstruct the center c_k of the hypercubic box that contains $x(t_k)$ from i_k. Simple calculation shows that

$$\|x(t_k) - c_k\| \leq \frac{1}{N} E_k, \quad \|c_k - x_k^*\| \leq \frac{N-1}{N} E_k. \tag{9}$$

The controller sets the control input $u(t) = K_{\sigma(t_k)} \hat{x}(t)$ for $t \in [t_k, t_{k+1})$, where $K_{\sigma(t_k)}$ is the state feedback gain matrix in Assumption 2, and $\hat{x}$ is the solution to the auxiliary system

$$\dot{\hat{x}} = A_{\sigma(t_k)} \hat{x} + B_{\sigma(t_k)} u = F_{\sigma(t_k)} \hat{x}, \qquad \hat{x}(t_k) = c_k. \tag{10}$$

(Notice that $\hat{x}$ is reset to c_k at each t_k.) Both the sensor and the controller maintain identical copies of the auxiliary system (10) to calculate the values x_{k+1}^* and E_{k+1} such that (8) holds at t_{k+1} on their own. The procedure of measuring the state and generating the control input is repeated for each $k \geq k_0$. The derivation of x_{k+1}^* and E_{k+1} from x_k^* and E_k is demonstrated in Subsections V-A and V-B.

V. Generating state bounds

In this section, we derive the values E_k. In Subsections V-A and V-B, we consider an arbitrary $k \geq k_0$ and establish x_{k+1}^* and E_{k+1} from x_k^* and E_k such that they satisfy

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq E_{k+1}. \tag{11}$$

The construction of E_k for $k \leq k_0$ and the existence of a k_0 such that (7) holds with $k = k_0$ are shown in Subsection V-C.

A. Sampling interval with no switch

In this subsection, we consider a $k \geq k_0$ such that

$$\sigma(t_k) = p = \sigma(t_{k+1}) \tag{12}$$

for some $p \in \mathcal{P}$. Then no switch has occurred on $(t_k, t_{k+1}]$ according to (4). Combining the switched linear system (1) and the auxiliary system (10) gives that

$$\dot{x} = A_p x + B_p u + D_p d,$$
$$\dot{\hat{x}} = A_p \hat{x} + B_p u.$$

Simple calculation shows that the error $e := x - \hat{x}$ satisfies

$$\|e(t_{k+1}^-)\| \le \frac{\Lambda_p}{N} E_k + \Phi_p(\tau_s)\delta =: E_{k+1} \tag{13}$$

with Λ_p defined in (5) and $\Phi_p : [0, \tau_s] \to \mathbb{R}$ defined as

$$\Phi_p(t) := \int_0^{\tau_s} \|e^{A_p s} D_p\| \mathrm{d}s. \tag{14}$$

By continuity, equation (11) holds with x_{k+1}^* defined as

$$x_{k+1}^* := \hat{x}(t_{k+1}^-) = e^{F_p \tau_s} c_k, \tag{15}$$

where $F_p = A_p + B_p K_p$ is defined in Assumption 2.

B. Sampling interval with a switch

In this subsection, we consider a $k \ge k_0$ such that

$$\sigma(t_k) = p \ne q = \sigma(t_{k+1}) \tag{16}$$

for some $p, q \in \mathcal{P}$. Then exactly one switch has occurred on $(t_k, t_{k+1}]$ according to (4). Denote the switching time by $t_k + \bar{t}$, where $\bar{t} \in (0, \tau_s]$ is unknown.

1) Before the switch: We proceed as in Subsection V-A and derive that the error $e = x - \hat{x}$ satisfies

$$\|e(t_k + \bar{t})\| \le \frac{\|e^{A_p \bar{t}}\|}{N} E_k + \Phi_p(\bar{t})\delta,$$

where Φ_p was defined in (14). As $t_k + \bar{t}$ is unknown, we estimate $x(t_k + \bar{t})$ by comparing it to $\hat{x}(t + t')$ for an arbitrary $t' \in [0, \tau_s)$ using the triangular inequality and derive

$$\|x(t_k + \bar{t}) - \hat{x}(t_k + t')\| \le \Phi_p(\bar{t})\delta + \|e^{F_p \bar{t}} - e^{F_p t'}\| \times \left(\|x_k^*\| + \frac{N-1}{N} E_k\right) + \frac{\|e^{A_p \bar{t}}\|}{N} E_k =: D_{k+1}(t', \bar{t}).$$

2) After the switch: Combining the switched linear system (1) and the auxiliary system (10) with $u = K_p \hat{x}$ gives that

$$\dot{z} = \bar{A}_{pq} z + \bar{D}_q d,$$

where $z := (x^\top, \hat{x}^\top)^\top \in \mathbb{R}^{2n_x}$ and

$$\bar{A}_{pq} := \begin{pmatrix} A_q & B_q K_p \\ 0_{n_x \times n_x} & A_p + B_p K_p \end{pmatrix}, \quad \bar{D}_q = \begin{pmatrix} D_q \\ 0_{n_x \times n_d} \end{pmatrix}.$$

Combining it with a second auxiliary system

$$\dot{\bar{z}} = \bar{A}_{pq} \bar{z}, \quad \bar{z}(t_k + t') = (\hat{x}(t_k + t')^\top, \hat{x}(t_k + t')^\top)^\top \tag{17}$$

gives

$$\dot{z} = \bar{A}_{pq} z + \bar{D}_q d, \qquad \dot{\bar{z}} = \bar{A}_{pq} \bar{z}$$

with the boundary condition

$$\|z(t_k + \bar{t}) - \bar{z}(t_k + t')\| \le D_{k+1}(t', \bar{t})$$

by the definition of ∞-norm. It is simple to derive that

$$\|z(t_{k+1}^-) - \bar{z}(t_{k+1} - \bar{t} + t')\| \le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| D_{k+1}(t', \bar{t}) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta,$$

where $\bar{\Phi}_{pq} : [0, \tau_s] \to \mathbb{R}$ is an increasing function defined as

$$\bar{\Phi}_{pq}(t) := \int_0^t \|e^{\bar{A}_{pq} s} \bar{D}_q\| \mathrm{d}s.$$

Similar to Subsubsection V-B.1, we estimate $z(t_{k+1}^-)$ by comparing it to $\bar{z}(t_k + t'')$ for an arbitrary $t'' \in [0, \tau_s)$ using the triangle inequality and derive

$$\begin{aligned} &\|z(t_{k+1}^-) - \bar{z}(t_k + t'')\| \\ &\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t'' - t')}\| \|e^{F_p t'}\| \left(\|x_k^*\| + \frac{N-1}{N} E_k\right) \\ &\quad + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| D_{k+1}(t', \bar{t}) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta \\ &=: \bar{E}_{k+1}(t', t'', \bar{t}) \end{aligned}$$

To eliminate the dependence on the unknown switching time $\bar{t}$, we take the maximum over $\bar{t}$ (with fixed t', t'') and define

$$E_{k+1} := \max\nolimits_{\bar{t} \in (0, \tau_s]} \bar{E}_{k+1}(t', t'', \bar{t}). \tag{18}$$

A relatively simple bound on E_{k+1} can be derived as

$$E_{k+1} \le \alpha_{2,pq} \|x_k^*\| + \beta_{2,pq} E_k + \gamma_{2,pq} \delta, \tag{19}$$

where

$$\begin{aligned} &\alpha_{2,pq} := e^{\|\bar{A}_{pq}\| \max\{\tau_s, 2(t'' - t'), \tau_s + 2(t' - t'')\}} \|\bar{A}_{pq}\| \\ &\quad \times \max\{t'' - t', \tau_s + t' - t''\} + \max\{\tau_s - t', t'\} \\ &\quad \times e^{\|\bar{A}_{pq}\| \tau_s} e^{\|A_p + B_p K_p\| \max\{\tau_s, 2t'\}} \|A_p + B_p K_p\|, \\ &\beta_{2,pq} := \frac{N-1}{N} \alpha_{2,pq} + \frac{1}{N} e^{(\|\bar{A}_{pq}\| + \|A_p\|)\tau_s}, \\ &\gamma_{2,pq} := e^{\|\bar{A}_{pq}\| \tau_s} \Phi_p(\tau_s) + \bar{\Phi}_{pq}(\tau_s). \end{aligned} \tag{20}$$

By continuity, equation (11) holds with x_{k+1}^* defined as

$$x_{k+1}^* := (I_{n_x} \ 0_{n_x}) \bar{z}(t_k + t'') = H_{pq} c_k, \tag{21}$$

where

$$H_{pq} := (I_{n_x} \ 0_{n_x}) e^{\bar{A}_{pq} t''} \begin{pmatrix} I_{n_x} \\ I_{n_x} \end{pmatrix} e^{F_p t'}.$$

C. Generating E_{k_0}

In the beginning, an arbitrary initial value E_0 is given. Before the state is captured, we calculate an increasing sequence $(E_k)_{k \ge 1}$ so that their growth rate dominates that of the state x under open-loop dynamics, and establish a $k_0 \in \mathbb{Z}_{\ge 0}$ such that (7) is satisfied. The sequence $(E_k)_{k \ge 1}$ is defined by the recursive formula

$$E_{k+1} = (1 + \epsilon_E) \hat{\Lambda}^2 E_k + (\hat{\Lambda} + 1)\Phi\delta, \tag{22}$$

where $\epsilon_E \in \mathbb{R}_{>0}$ can be arbitrarily small, and

$$\hat{\Lambda} := \max_{p \in \mathcal{P}} \max_{t \in [0, \tau_s]} \|e^{A_p t}\| \ge 1, \quad \Phi := \max_{p \in \mathcal{P}} \Phi_p(\tau_s), \tag{23}$$

and Φ_p is defined according to (14) for each p. We proceed to show that, for each $E_0 \in \mathbb{R}_{>0}$, there exists a $k_0 \in \mathbb{Z}_{\ge 0}$ such that (7) holds with $k = k_0$. Indeed, consider a $k \in Z_{\ge 0}$ such that $\|x(t_k)\| > E_k$. Then $u(t) = 0$ for all $t \in [t_k, t_{k+1})$ by Subsection IV-A, and simply calculation shows that

$$\|x(t)\| \le \hat{\Lambda}^2 \|x(t_k)\| + (\hat{\Lambda} + 1)\Phi\delta \qquad \forall\, t \in (t_k, t_{k+1}]. \tag{24}$$

If $\|x_0\| \le E_0$, let $k_0 = 0$. Otherwise, let $k \in \mathbb{Z}_{>0}$ be such that $\|x(t_l)\| > E_l$ for all $l < k$. Consider the *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ defined as $\lceil r \rceil := \min\{j \in \mathbb{Z} : j \ge r\}$, and let

$$\bar{k} := \left\lceil \frac{\log \|x_0\| - \log E_0}{\log(1 + \epsilon_E)} \right\rceil \in \mathbb{Z}_{\ge 0}.$$

Then $\bar{k} \geq 1$ as $\|x_0\| > E_0$, and (22) and (24) imply that

$$E_{\bar{k}} \geq \hat{\Lambda}^{2\bar{k}}\|x_0\| + \sum_{l=0}^{2\bar{k}-1} \hat{\Lambda}^l \Phi\delta \geq \|x(t_{\bar{k}})\|.$$

Thus there exists at least one $k \in \mathbb{Z}_{\geq 0}$ such that $\|x(t_k)\| \leq E_k$. Then $k_0 := \min\{k \in \mathbb{Z}_{\geq 0} : \|x(t_k)\| \leq E_k\}$ satisfies

$$k_0 \leq \eta(\|x_0\|), \tag{25}$$

where $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as

$$\eta(r) := \max\left\{0, \left\lceil \frac{\log r - \log E_0}{\log(1+\epsilon_E)} \right\rceil \right\}. \tag{26}$$

Moreover, (22), (24) and Young's inequality imply that

$$\begin{aligned} \|x(t)\| &\leq \gamma(\|x_0\|) + \frac{1}{\kappa}\Phi^\kappa \delta^\kappa \qquad \forall\, t \in [0, t_{k_0}], \\ E_{k_0} &\leq \gamma(\|x_0\|) + \frac{1}{\kappa}\Phi^\kappa \delta^\kappa, \end{aligned} \tag{27}$$

where $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is an increasing function defined as

$$\begin{aligned} \gamma(r) &:= (1+\epsilon_E)^{\eta(r)} \hat{\Lambda}^{2\eta(r)} E_0 \\ &\quad + \frac{1}{\varsigma}\left(\frac{(1+\epsilon_E)^{\eta(r)}\hat{\Lambda}^{2\eta(r)} - 1}{(1+\epsilon_E)\hat{\Lambda}^2 - 1} (\hat{\Lambda}+1) \right)^\varsigma, \end{aligned} \tag{28}$$

with arbitrary $\kappa, \varsigma \in (1, \infty)$ such that $1/\kappa + 1/\varsigma = 1$.

VI. Stability analysis

In this section we show that the communication and control strategy described in Section IV fulfills Theorem 1. The proof details are omitted due to space constraints.

A. Sampling interval with no switch

In this subsection, we consider a $k \geq k_0$ satisfying (12) with some $p \in \mathcal{P}$. Consider $S_p := e^{(A_p + B_p K_p)\tau_s}$. As $A_p + B_p K_p$ is Hurwitz, there exist $P_p, Q_p > 0$ such that

$$S_p^\top P_p S_p - P_p = -Q_p < 0.$$

Define

$$\chi_p := \frac{2n^2 \|S_p^\top P_p S_p\|^2}{\underline{\lambda}(Q_p)} + n\|S_p^\top P_p S_p\|. \tag{29}$$

By Assumption 4, there exist a constant $\rho_p \in \mathbb{R}_{>0}$ for each p and a constant $\psi_1 \in \mathbb{R}_{>0}$ such that

$$\frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p} + (1+\psi_1)\frac{\Lambda_p^2}{N^2} < 1.$$

Define a function $V_r : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for each $r \in \mathcal{P}$ as

$$V_r(x, E) := x^\top P_r x + \rho_r E^2. \tag{30}$$

Lemma 1. *For all $k \geq k_0$ such that* (12) *holds, the function V_p defined according to* (30) *satisfies*

$$V_p(x_{k+1}^*, E_{k+1}) \leq \nu V_p(x_k^*, E_k) + \nu_d \delta^2, \tag{31}$$

where

$$\begin{aligned} \nu &:= \max\nolimits_{p\in\mathcal{P}} \nu_p, \\ \nu_p &:= \max\left\{ \frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p} + (1+\psi_1)\frac{\Lambda_p^2}{N^2}, 1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)} \right\}, \\ \nu_d &:= \max\nolimits_{p\in\mathcal{P}} \left(1 + \frac{1}{\psi_1}\right)\rho_p \Phi_p(\tau_s)^2, \end{aligned}$$

and Λ_p, Φ_p, χ_p are defined in (5), (14), (29), *respectively.*

B. Sampling interval with a switch

In this subsection, we consider a $k \geq k_0$ satisfying (16) with some $p, q \in \mathcal{P}$. Let $h_{pq} := \sqrt{\overline{\lambda}(H_{pq}^\top H_{pq})}$.

Lemma 2. *For all $k \geq k_0$ such that* (16) *holds, the functions V_p, V_q defined according to* (30) *satisfy*

$$V_q(x_{k+1}^*, E_{k+1}) \leq \mu V_p(x_k^*, E_k) + \mu_d \delta^2, \tag{32}$$

where

$$\begin{aligned} \mu &:= \max\nolimits_{p,q\in\mathcal{P}} \mu_{pq}, \\ \mu_{pq} &:= \max\left\{ \frac{2\overline{\lambda}(P_q)h_{pq}^2}{\underline{\lambda}(P_p)} + (2+\psi_2)\frac{\alpha_{2,pq}^2 \rho_q}{\underline{\lambda}(P_p)}, \right. \\ &\qquad \left. \frac{2n\overline{\lambda}(P_q)h_{pq}^2}{\rho_p}\frac{(N-1)^2}{N^2} + (2+\psi_2)\frac{\beta_{2,pq}^2\rho_q}{\rho_p} \right\}, \\ \mu_d &:= \max\nolimits_{p,q\in\mathcal{P}} \left(1 + \frac{2}{\psi_2}\right)\rho_q \gamma_{2,pq}^2, \end{aligned}$$

$\alpha_{2,pq}$, $\beta_{2,pq}$, $\gamma_{2,pq}$ are defined in (20), *$\psi_2 \in \mathbb{R}_{>0}$ is arbitrary.*

Remark 1. The definition of ν in Lemma 1 implies $\nu < 1$. Moreover, the definition of μ in Lemma 2 implies $\mu \geq 1$ when $t' = t'' = 0$. For general t' and t'', we can guarantee this by letting $\mu = \max\{\mu, 1\}$ if necessary. On the other hand, as (32) holds for all $\psi_2 \in \mathbb{R}_{>0}$, a sufficiently small ψ_2 (for a fixed ψ_1) can be selected so that $\mu_d \geq \nu_d$. We assume that $\mu \geq 1 > \nu$ and $\mu_d \geq \nu_d$ in the following proof.

C. Combined bound for sampling times

In this subsection, we give a lower bound on the average dwell-time τ_a in Assumption 1 that guarantees convergence.

Lemma 3. *Consider ν, ν_d, μ and μ_d defined in Lemmas 1 and 2. If the average dwell-time τ_a satisfies*

$$\tau_a > \left(1 + \frac{\log \mu}{\log(1/\nu)}\right)\tau_s, \tag{33}$$

then there exists a constant $\phi \in (0,1)$ such that

$$\begin{aligned} V_{\sigma(t_k)}(x_k^*, E_k) &< \theta^{k-k_0}\Theta^{N_0} V_{\sigma(t_{k_0})}(0, E_{k_0}) \\ &\quad + \Theta^{N_0+1}\left(1 + \frac{\nu}{\phi(1-\nu)}\right)\nu_d\delta^2 \end{aligned} \tag{34}$$

for all $k \geq k_0$, where N_0 is defined in (2), *and*

$$\begin{aligned} \Theta &:= \frac{\mu + \phi(1-\nu)\mu_d/\nu_d}{\nu + \phi(1-\nu)} > 1, \\ \theta &:= \Theta^{\tau_s/\tau_a}(\nu + \phi(1-\nu)) < 1. \end{aligned} \tag{35}$$

D. Inter-sample bound and exponential convergence

In this subsection, we describe an over-approximation of the state between sampling times and establish the first claim of Theorem 1. Similar analysis to Subsection V-B shows that

$$\|x(t)\| \leq (\alpha_{2,pq}+1)\|x_k^*\| + \left(\beta_{2,pq} + \frac{N-1}{N}\right)E_k + \gamma_{2,pq}\delta,$$

for all $k \geq k_0$ and all $t \in (t_k, t_{k+1}]$. Combining this inequality with (34) gives that

$$\|x(t)\| \leq \theta^{(k-k_0)/2}\bar{c}E_{k_0} + \bar{d}\delta \qquad \forall\, t \in (t_{k_0}, \infty), \tag{36}$$

where

$$\bar{c} := \Theta^{N_0/2}\Bigg(\bigg(\max_{p,q\in\mathcal{P}}\alpha_{2,pq}+1\bigg)\sqrt{\frac{\max_{p\in\mathcal{P}}\rho_p}{\min_{p\in\mathcal{P}}\underline{\lambda}(P_p)}} + \bigg(\max_{p,q\in\mathcal{P}}\beta_{2,pq}+\frac{N-1}{N}\bigg)\sqrt{\frac{\max_{p\in\mathcal{P}}\rho_p}{\min_{p\in\mathcal{P}}\rho_p}}\Bigg), \quad (37)$$

$$\bar{d} := \Bigg(\bigg(\max_{p,q\in\mathcal{P}}\alpha_{2,pq}+1\bigg)\frac{1}{\sqrt{\min_{p\in\mathcal{P}}\underline{\lambda}(P_p)}} + \bigg(\max_{p,q\in\mathcal{P}}\beta_{2,pq}+\frac{N-1}{N}\bigg)\frac{1}{\sqrt{\min_{p\in\mathcal{P}}\rho_p}}\Bigg) \times \Theta^{(N_0+1)/2}\sqrt{\bigg(1+\frac{\nu}{\phi(1-\nu)}\bigg)\nu_d} + \gamma_2. \quad (38)$$

Combining (25), (27) and (36) gives (6) with

$$\lambda := -\frac{\log\theta}{2\tau_s} > 0, \quad (39)$$

$$g(r) := \theta^{-(\eta(r)+1)/2}\bar{c}\gamma(r), \quad (40)$$

$$h(r) := \frac{1}{\kappa}\theta^{-1/2}\bar{c}\Phi^{\kappa}r^{\kappa} + \bar{d}r, \quad (41)$$

where Φ, η, γ, θ, $\bar{c}$ and $\bar{d}$ are defined in (23), (26), (28), (35), (37) and (38), respectively, and $\kappa \in (1,\infty)$ is arbitrary.

E. BIBS stability

In this subsection, we provide a sufficient condition for the second claim of Theorem 1, which essentially follows from similar analysis to previous subsections and [21, Subsection 5.5]. Compared with [21, Subsection 5.5], an explicit bound on the average dwell-time guaranteeing BIBS stability (42) is provided, and its relation with the bound guaranteeing exponential convergence (33) is discussed in Remark 2.

Lemma 4. *If the average dwell-time τ_a satisfies*

$$\tau_a > \left(1+\frac{\log\beta_2}{\log(N/\Lambda)}\right)\tau_s, \quad (42)$$

where

$$\Lambda := \max\nolimits_{p\in\mathcal{P}}\Lambda_p, \qquad \beta_2 := \max\nolimits_{p,q\in\mathcal{P}}\beta_{2,pq},$$

then for each $\epsilon \in \mathbb{R}_{>0}$, there exists a disturbance bound $\delta \in \mathbb{R}_{>0}$ such that if, in addition to (3)*, the initial state satisfies $\|x_0\| \le \delta$, then $\|x(t)\| \le \epsilon$ for all $t \in \mathbb{R}_{\ge 0}$.*

Remark 2. We may replace ρ_p with $\rho = \max_{p\in\mathcal{P}}\rho_p$ in the definition (30) and let the rest of the analysis remain unchanged. We will show that (33) implies (42) under such modification. Indeed, the definitions of ν and μ guarantee that $1 > \nu > \Lambda^2/N^2$ and $\mu > \beta_2^2$, which further imply that $(\log\mu)/(\log(1/\nu)) > (\log\beta_2)(\log(N/\Lambda))$.

VII. Conclusion

We presented a result on the stabilization of a switched linear system with disturbance using sampled-data quantized feedback. BIBS stability and exponential convergence with respect to the initial state were established via propagating over-approximations of reachable sets. Compared with earlier results, the approximation bounds were enlarged to handle the disturbance. Future work will be focusing on relaxing the assumption on the known bound on the disturbance.

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