On Topological Entropy of Switched Linear Systems with Pairwise Commuting Matrices

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Motivation

Topological entropy in systems theory

- Originated from [Kolmogorov, 1958], defined by [Adler, Konheim, and McAndrew, 1965], [Bowen, 1971], and [Dinaburg, 1970].

- Essential idea:
  - The complexity growth of a system.
  - The information accumulation needed to approximate a trajectory.

- In control theory:
  - Topological feedback entropy [Nair, Evans, Mareels, and Moran, 2004]
  - Invariance entropy [Colonius and Kawan, 2009], exponential stabilization entropy [Colonius, 2012]
  - Estimation entropy [Savkin, 2006] and [Liberzon and Mitra, 2018]

- Minimal data rate for stabilizing linear time-invariant (LTI) system [Hespanha, Ortega, and Vasudevan, 2002], [Nair and Evans, 2003], and [Tatikonda and Mitter, 2004]
Motivation

Switched linear system with pairwise commuting matrices

- Switching is ubiquitous in realistic system models.
- Stability under arbitrary switching: pairwise commuting matrices
  [Narendra and Balakrishnan, 1994]
- Neither minimal data rate for stabilization nor topological entropy is well-understood:
  - Sufficient data rate [Liberzon, 2014], [Yang and Liberzon, 2018], and [Sibai and Mitra, 2017]
  - Topological entropy [Yang, Schmidt, and Liberzon, 2018]
Switched System

A finite family of continuous-time dynamical systems

\[ \dot{x} = f_p(x), \quad p \in \mathcal{P} \]

with the state \( x \in \mathbb{R}^n \) and an index set \( \mathcal{P} \).

A switched system

\[ \dot{x} = f_\sigma(x), \quad x(0) \in K. \]

- Switching signal \( \sigma : \mathbb{R}_+ \to \mathcal{P} \) is right-continuous and piecewise constant
- Initial set \( K \) is compact with a nonempty interior
- Modes \( \{f_p : p \in \mathcal{P}\} \)
- Denote by \( \xi_\sigma(x, t) \) the solution at \( t \) with switching signal \( \sigma \) and initial state \( x \)
Entropy Definition

A switched system
\[ \dot{x} = f_\sigma(x), \quad x(0) \in K. \]

- Given a time horizon \( T \geq 0 \) and a radius \( \varepsilon > 0 \), define the open ball:
\[ B_{f_\sigma}(x, \varepsilon, T) := \left\{ x' \in K : \max_{t \in [0, T]} \| \xi_\sigma(x', t) - \xi_\sigma(x, t) \| < \varepsilon \right\}. \]

- A finite set \( E \subset K \) is \((T, \varepsilon)\)-spanning if \( K = \bigcup_{\hat{x} \in E} B_{f_\sigma}(\hat{x}, \varepsilon, T) \).

- Let \( S(f_\sigma, \varepsilon, T, K) \) be the minimal cardinality of a \((T, \varepsilon)\)-spanning set.

- The topological entropy with initial set \( K \) and switching signal \( \sigma \) is defined in terms of the exponential growth rate of \( S(f_\sigma, \varepsilon, T, K) \) by
\[ h(f_\sigma, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K). \]
Alternative Entropy Definition

A switched system
\[ \dot{x} = f_\sigma(x), \quad x(0) \in K. \]

- The topological entropy is defined by

\[
h(f_\sigma, K) := \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K).\]

- A finite set of points \( E \subset K \) is \((T, \varepsilon)\)-separated if for all \( \hat{x}, \hat{x}' \in E \),

\[
\hat{x}' \notin B_{f_\sigma}(\hat{x}, \varepsilon, T) = \left\{ x \in K : \max_{t \in [0,T]} \| \xi_\sigma(x, t) - \xi_\sigma(\hat{x}, t) \| < \varepsilon \right\}.
\]

- Let \( N(f_\sigma, \varepsilon, T, K) \) be the maximal cardinality of a \((T, \varepsilon)\)-separated set.

- Proposition 1. The topological entropy satisfies

\[
h(f_\sigma, K) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log N(f_\sigma, \varepsilon, T, K).
\]

Proof. \( N(f_\sigma, 2\varepsilon, T, K) \leq S(f_\sigma, \varepsilon, T, K) \leq N(f_\sigma, \varepsilon, T, K).\)
Active Time and Active Rates

For a switching signal $\sigma$, define the following quantities.

- The active time of each mode over an interval $[0, t]$ is

$$\tau_p(t) := \int_0^t 1_p(\sigma(s)) \, ds, \quad p \in \mathcal{P}$$

with the indicator function $1$. Then $\sum_{p \in \mathcal{P}} \tau_p(t) = t$.

- The active rate of each mode over $[0, t]$ is

$$\rho_p(t) := \tau_p(t)/t, \quad p \in \mathcal{P}$$

with $\rho_p(0) := 1_p(\sigma(0))$. Then $\sum_{p \in \mathcal{P}} \rho_p(t) = 1$.

- The asymptotic active rate of each mode is

$$\hat{\rho}_p := \limsup_{t \to \infty} \rho_p(t), \quad p \in \mathcal{P}.$$  

It is possible that $\sum_{p \in \mathcal{P}} \hat{\rho}_p > 1$. 
Entropy of Switched Linear Systems

A switched linear system

\[ \dot{x} = A_\sigma x, \quad x(0) \in K. \]

Results from [Yang, Schmidt, and Liberzon, 2018]:

- **Proposition 2.** The topological entropy of the switched linear system is independent of the choice of the initial set \( K \).
- **Proposition 3.** The topological entropy of the switched linear system satisfies

\[
\limsup_{t \to \infty} \sum_{p \in \mathcal{P}} \text{tr}(A_p) \rho_p(t) \leq h(A_\sigma) \leq \limsup_{t \to \infty} \sum_{p \in \mathcal{P}} n\|A_p\| \rho_p(t)
\]

with the active rates \( \rho_p \).

**Proof for the upper bound.**

1. The solutions satisfy 

\[
\|\xi_\sigma(x', t) - \xi_\sigma(x, t)\| \leq e \sum_p \|A_p\| \tau_p(t) \|x' - x\|
\]

2. Construct a \((T, \varepsilon)\)-spanning set using a grid.

Lack of “independence” between eigenspaces of different modes!
Switched Commuting Linear Systems

A switched linear system

\[ \dot{x} = A_\sigma x, \quad x(0) \in K \]

with a commuting family \( \{A_p : p \in \mathcal{P}\} \).

- If all \( A_p \) are diagonalizable, then there is a change of basis under which all \( A_p \) are diagonal.
- Every scalar component evolves independently (under the same switching signal).
- A formula for the entropy was established in [Yang, Schmidt, and Liberzon, 2018].
Switched Commuting Linear Systems

A switched linear system

\[ \dot{x} = A_\sigma x, \quad x(0) \in K \]

with a commuting family \( \{A_p : p \in \mathcal{P}\} \).

- A well-known result: in general, there is a change of basis under which all \( A_p \) are upper triangular.
- Each scalar component evolves in a “strict-feedback” fashion.
- An upper bound for the entropy was established in [Yang, Schmidt, and Liberzon, 2018].
- Being simultaneously triangularizable is weaker than being pairwise commuting!
A switched linear system

\[ \dot{x} = A_\sigma x, \quad x(0) \in K \]

with a commuting family \( \{ A_p : p \in \mathcal{P} \} \).

- First goal: a suitable change of basis for pairwise commuting matrices.
- Jordan–Chevalley Decomposition [Humphreys, 1972]. For each matrix \( A \), there exist polynomials \( f \) and \( g \), without constant term, such that \( f(A) \) is a diagonalizable matrix, \( g(A) \) is a nilpotent matrix, and

\[ A = f(A) + g(A). \]

- A polynomial of a matrix \( A \) commutes with all matrices that commute with \( A \).
A Change of Basis

Proposition 5

For the commuting family \( \{A_p : p \in \mathcal{P}\} \), there exists an invertible matrix \( \Gamma \in \mathbb{C}^{n \times n} \) such that

\[
\Gamma A_p \Gamma^{-1} = D_p + N_p \quad \forall p \in \mathcal{P},
\]

where all \( D_p \in \mathbb{C}^{n \times n} \) are diagonal matrices, all \( N_p \in \mathbb{C}^{n \times n} \) are nilpotent matrices, and \( \{D_p, N_p : p \in \mathcal{P}\} \) is a commuting family.

Proof.

1. For each \( p \), there are polynomials \( f_p \) and \( g_p \) such that \( f_p(A_p) \) is diagonalizable, \( g_p(A_p) \) is nilpotent, and \( A_p = f_p(A_p) + g_p(A_p) \).
2. The set \( \{f_p(A_p), g_p(A_p) : p \in \mathcal{P}\} \) is a commuting family.
3. There is an invertible \( \Gamma \in \mathbb{C}^{n \times n} \) such that all \( D_p := \Gamma f_p(A_p) \Gamma^{-1} \) are invertible.
4. All \( N_p := \Gamma g_p(A_p) \Gamma^{-1} \) are nilpotent, and \( \{D_p, N_p : p \in \mathcal{P}\} \) is a commuting family. \( \square \)
A Formula for Entropy

The switched commuting linear system becomes

\[ \dot{x} = (D_\sigma + N_\sigma)x, \quad x(0) \in K, \]

where \( D_p = \text{diag}(a^1_p, \ldots, a^n_p) \) are diagonal, \( N_p \) are nilpotent, and \( \{D_p, N_p : p \in \mathcal{P}\} \) is a commuting family.

**Theorem 6**

The topological entropy of the switched commuting system satisfies

\[ h(D_\sigma + N_\sigma) = \lim \sup_{T \to \infty} \sum_{i=1}^n \frac{1}{T} \max_{t \in [0, T]} \sum_{p \in \mathcal{P}} \text{Re}(a^i_p) \tau_p(t) \]

with the active times \( \tau_p \).

- The entropy only depends on the diagonal part, i.e.,
  \[ h(D_\sigma + N_\sigma) = h(D_\sigma). \]
A Formula for Entropy

Proof.

1. The solutions satisfy

\[ \| \xi_\sigma(x', t) - \xi_\sigma(x, t) \| = \left\| e^{ \sum_{p \in P} N_p \tau_p(t) } e^{ \sum_{p \in P} D_p \tau_p(t) } (x' - x) \right\|. \]

2. Lemma 2. Consider the commuting family of nilpotent matrices \( \{ N_p : p \in P \} \).
   For each \( \delta > 0 \), there is a constant \( c_\delta > 0 \) such that for all \( v \in \mathbb{C}^n \),

\[ c_\delta^{-1} e^{-\delta t} \| v \| \leq \left\| e^{ \sum_{p \in P} N_p \tau_p(t) } v \right\| \leq c_\delta e^{\delta t} \| v \| \]

for all \( t \geq 0 \) with the active times \( \tau_p \).

3. Given a radius \( \varepsilon > 0 \), there is a constant \( c_\varepsilon > 0 \) such that

\[ \cdots \leq \| \xi_\sigma(x', t) - \xi_\sigma(x, t) \| \leq c_\varepsilon e^{\varepsilon t} \max_{i=1, \ldots, n} e^{ \sum_{p \in P} \Re(a_p^i \tau_p(t)) } |x'_i - x_i|. \]

4. For the upper/lower bound, construct a \((T, \varepsilon)\)-spanning/separated set using a grid. \qed
The Non-Switched Case

The formula for entropy yields the following well-known result [Bowen, 1971]:

**Corollary 7**

The topological entropy of the linear time-invariant (LTI) system

\[ \dot{x} = Ax, \quad x(0) \in K \]

equals the sum of the positive real parts of the eigenvalues of \( A \), that is,

\[ h(A) = \sum_{\lambda \in \text{spec}(A)} \max\{\text{Re}(\lambda), 0\}. \]

**Proof.**

1. The spectrum \( \text{spec}(A) = \{a^1, \ldots, a^n\} \).
2. The entropy

\[ h(A) = \limsup_{T \to \infty} \frac{1}{T} \sum_{i=1}^{n} \max_{t \in [0,T]} \text{Re}(a^i)t = \sum_{i=1}^{n} \max\{\text{Re}(a^i), 0\}. \]
More General Upper and Lower Bounds for Entropy

<table>
<thead>
<tr>
<th>#</th>
<th>Formula/upper bounds</th>
<th>Sw</th>
<th>CoB</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$= \limsup_{T \to \infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t \in [0,T]} \sum_{p \in \mathcal{P}} \Re(a_i^p) \tau_p(t)$</td>
<td>$\tau_p$</td>
<td>Yes</td>
</tr>
<tr>
<td>(2)</td>
<td>$\leq \sum_{i=1}^{n} \max \left{ \limsup_{t \to \infty} \sum_{p \in \mathcal{P}} \Re(a_i^p) \rho_p(t), 0 \right}$</td>
<td>$\rho_p$</td>
<td>Yes</td>
</tr>
<tr>
<td>(3)</td>
<td>$\leq \limsup_{t \to \infty} \sum_{p \in \mathcal{P}} h(D_p) \rho_p(t)$</td>
<td>$\rho_p$</td>
<td>No</td>
</tr>
<tr>
<td>(4)</td>
<td>$\leq \sum_{p \in \mathcal{P}} h(D_p) \hat{\rho}_p$</td>
<td>$\hat{\rho}_p$</td>
<td>No</td>
</tr>
<tr>
<td>(5)</td>
<td>$\leq \max_{p \in \mathcal{P}} h(D_p)$</td>
<td>N/A</td>
<td>No</td>
</tr>
<tr>
<td>(6)</td>
<td>$\leq \limsup_{t \to \infty} \sum_{p \in \mathcal{P}} n|A_p| \rho_p(t)$</td>
<td>$\rho_p$</td>
<td>No</td>
</tr>
<tr>
<td>(7)</td>
<td>$\geq \limsup_{t \to \infty} \sum_{p \in \mathcal{P}} \text{tr}(A_p) \rho_p(t)$</td>
<td>$\rho_p$</td>
<td>No</td>
</tr>
</tbody>
</table>
Numerical Example

Let $\mathcal{P} = \{1, 2\}$ and

$$D_1 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

<table>
<thead>
<tr>
<th></th>
<th>$(\hat{\rho}_1, \hat{\rho}_2)$</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No switch</td>
<td>$(1, 0)$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Periodic switches</td>
<td>$(0.5, 0.5)$</td>
<td>2</td>
<td>2</td>
<td>2.5</td>
<td>2.5</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Switches w/ set-points</td>
<td>$(0.9, 0.9)$</td>
<td>2.8</td>
<td>4.4</td>
<td>2.9</td>
<td>4.5</td>
<td>3</td>
<td>5.8</td>
<td>2.8</td>
</tr>
</tbody>
</table>
Conclusion

Contributions:

- Switched linear systems with pairwise commuting matrices.
- A change of basis under which each of the matrices can be decomposed into a diagonal part and a nilpotent part, and all the diagonal and nilpotent parts are pairwise commuting.
- A formula for the topological entropy, which only depends on the diagonal part.
- More general upper bounds for the entropy.

Future research:

- Reconcile the switching characterizations for entropy computation and for stability analysis and control design
  - Stability and stabilization: slow-switching conditions such as the average dwell-time
  - Entropy: the active time (rarely seen in the literature)


References II


