On Topological Entropy of Switched Linear Systems with Pairwise Commuting Matrices

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Motivation

Topological entropy in systems theory

- Originated from [Kolmogorov, 1958], defined by [Adler, Konheim, and McAndrew, 1965], [Bowen, 1971], and [Dinaburg, 1970].
- Essential idea:
  - The complexity growth of a system.
  - The information accumulation needed to approximate a trajectory.
- In control theory:
  - Topological feedback entropy [Nair, Evans, Mareels, and Moran, 2004]
  - Invariance entropy [Colonius and Kawan, 2009], exponential stabilization entropy [Colonius, 2012]
  - Estimation entropy [Savkin, 2006] and [Liberzon and Mitra, 2018]
- Minimal data rate for stabilizing linear time-invariant (LTI) system [Hespanha, Ortega, and Vasudevan, 2002], [Nair and Evans, 2003], and [Tatikonda and Mitter, 2004]

Motivation

Switched linear system with pairwise commuting matrices

- Switching is ubiquitous in realistic system models.
- Stability under arbitrary switching: pairwise commuting matrices [Narendra and Balakrishnan, 1994]
- Neither minimal data rate for stabilization nor topological entropy is well-understood:
  - Sufficient data rate [Liberzon, 2014], [Yang and Liberzon, 2018], and [Sibai and Mitra, 2017]
  - Topological entropy [Yang, Schmidt, and Liberzon, 2018]

Switched System

A finite family of continuous-time dynamical systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P}$$

with the state $x \in \mathbb{R}^n$ and an index set $\mathcal{P}$.

A switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K.$$

- Switching signal $\sigma : \mathbb{R}_+ \to \mathcal{P}$ is right-continuous and piecewise constant
- Initial set K is compact with a nonempty interior
- Modes $\{f_p : p \in \mathcal{P}\}$
- Denote by $\xi_\sigma(x, t)$ the solution at t with switching signal σ and initial state x

Entropy Definition

A switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K.$$

- Given a time horizon $T \geq 0$ and a radius $\varepsilon > 0$, define the open ball:

$$B_{f_\sigma}(x, \varepsilon, T) := \left\{ x' \in K : \max_{t \in [0,T]} \|\xi_\sigma(x', t) - \xi_\sigma(x, t)\| < \varepsilon \right\}.$$

- A finite set $E \subset K$ is (T, ε)-spanning if $K = \bigcup_{\hat{x} \in E} B_{f_\sigma}(\hat{x}, \varepsilon, T)$.
- Let $S(f_\sigma, \varepsilon, T, K)$ be the minimal cardinality of a (T, ε)-spanning set.
- The topological entropy with initial set K and switching signal σ is defined in terms of the exponential growth rate of $S(f_\sigma, \varepsilon, T, K)$ by

$$h(f_\sigma, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K).$$

Alternative Entropy Definition

A switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K.$$

- The topological entropy is defined by

$$h(f_\sigma, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T, K).$$

- A finite set of points $E \subset K$ is (T, ε)-separated if for all $\hat{x}, \hat{x}' \in E$,

$$\hat{x}' \notin B_{f_\sigma}(\hat{x}, \varepsilon, T) = \left\{ x \in K : \max_{t \in [0,T]} \|\xi_\sigma(x,t) - \xi_\sigma(\hat{x},t)\| < \varepsilon \right\}.$$

- Let $N(f_\sigma, \varepsilon, T, K)$ be the maximal cardinality of a (T, ε)-separated set.
- Proposition 1. The topological entropy satisfies

$$h(f_\sigma, K) = \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log N(f_\sigma, \varepsilon, T, K).$$

Proof. $N(f_\sigma, 2\varepsilon, T, K) \leq S(f_\sigma, \varepsilon, T, K) \leq N(f_\sigma, \varepsilon, T, K)$. $\square$

Active Time and Active Rates

For a switching signal σ, define the following quantities.

- The active time of each mode over an interval $[0, t]$ is

$$\tau_p(t) := \int_0^t \mathbb{1}_p(\sigma(s)) \, \mathrm{d}s, \qquad p \in \mathcal{P}$$

  with the indicator function $\mathbb{1}$. Then $\sum_{p \in \mathcal{P}} \tau_p(t) = t$.

- The active rate of each mode over $[0, t]$ is

$$\rho_p(t) := \tau_p(t)/t, \qquad p \in \mathcal{P}$$

  with $\rho_p(0) := \mathbb{1}_p(\sigma(0))$. Then $\sum_{p \in \mathcal{P}} \rho_p(t) = 1$.

- The asymptotic active rate of each mode is

$$\hat{\rho}_p := \limsup_{t \to \infty} \rho_p(t), \qquad p \in \mathcal{P}.$$

  It is possible that $\sum_{p \in \mathcal{P}} \hat{\rho}_p > 1$.

Entropy of Switched Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K.$$

Results from [Yang, Schmidt, and Liberzon, 2018]:

- Proposition 2. The topological entropy of the switched linear system is independent of the choice of the initial set K.
- Proposition 3. The topological entropy of the switched linear system satisfies

$$\limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(A_p)\rho_p(t) \le h(A_\sigma) \le \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} n\|A_p\|\rho_p(t)$$

  with the active rates ρ_p.

  Proof for the upper bound.

  1. The solutions satisfy $\|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| \le e^{\sum_p \|A_p\|\tau_p(t)}\|x' - x\|$.
  2. Construct a (T,ε)-spanning set using a grid. □

  Lack of "independence" between eigenspaces of different modes!

Switched Commuting Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K$$

with a commuting family $\{A_p : p \in \mathcal{P}\}$.

- If all A_p are diagonalizable, then there is a change of basis under which all A_p are diagonal.
- Every scalar component evolves independently (under the same switching signal).
- A formula for the entropy was established in [Yang, Schmidt, and Liberzon, 2018].

Switched Commuting Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K$$

with a commuting family $\{A_p : p \in \mathcal{P}\}$.

- A well-known result: in general, there is a change of basis under which all A_p are upper triangular.
- Each scalar component evolves in a "strict-feedback" fashion.
- An upper bound for the entropy was established in [Yang, Schmidt, and Liberzon, 2018].
- Being simultaneously triangularizable is weaker than being pairwise commuting!

Switched Commuting Linear Systems

A switched linear system

$$\dot{x} = A_\sigma x, \qquad x(0) \in K$$

with a commuting family $\{A_p : p \in \mathcal{P}\}$.

- First goal: a suitable change of basis for pairwise commuting matrices.
- Jordan–Chevalley Decomposition [Humphreys, 1972]. For each matrix A, there exist polynomials f and g, without constant term, such that $f(A)$ is a diagonalizable matrix, $g(A)$ is a nilpotent matrix, and

$$A = f(A) + g(A).$$

- A polynomial of a matrix A commutes with all matrices that commute with A.

A Change of Basis

Proposition 5

For the commuting family $\{A_p : p \in \mathcal{P}\}$, there exists an invertible matrix $\Gamma \in \mathbb{C}^{n\times n}$ such that

$$\Gamma A_p \Gamma^{-1} = D_p + N_p \qquad \forall\, p \in \mathcal{P},$$

where all $D_p \in \mathbb{C}^{n\times n}$ are diagonal matrices, all $N_p \in \mathbb{C}^{n\times n}$ are nilpotent matrices, and $\{D_p, N_p : p \in \mathcal{P}\}$ is a commuting family.

Proof.

1. For each p, there are polynomials f_p and g_p such that $f_p(A_p)$ is diagonalizable, $g_p(A_p)$ is nilpotent, and $A_p = f_p(A_p) + g_p(A_p)$.
2. The set $\{f_p(A_p), g_p(A_p) : p \in \mathcal{P}\}$ is a commuting family.
3. There is an invertible $\Gamma \in \mathbb{C}^{n\times n}$ such that all $D_p := \Gamma f_p(A_p)\Gamma^{-1}$ are invertible.
4. All $N_p := \Gamma g_p(A_p)\Gamma^{-1}$ are nilpotent, and $\{D_p, N_p : p \in \mathcal{P}\}$ is a commuting family. □

A Formula for Entropy

The switched commuting linear system becomes

$$\dot{x} = (D_\sigma + N_\sigma)x, \qquad x(0) \in K,$$

where $D_p = \mathrm{diag}(a_p^1, \ldots, a_p^n)$ are diagonal, N_p are nilpotent, and $\{D_p, N_p : p \in \mathcal{P}\}$ is a commuting family.

Theorem 6

The topological entropy of the switched commuting system satisfies

$$h(D_\sigma + N_\sigma) = \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t)$$

with the active times τ_p.

- The entropy only depends on the diagonal part, i.e., $h(D_\sigma + N_\sigma) = h(D_\sigma)$.

A Formula for Entropy

Proof.

1. The solutions satisfy

$$\|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| = \left\| e^{\sum_{p\in\mathcal{P}} N_p \tau_p(t)} e^{\sum_{p\in\mathcal{P}} D_p \tau_p(t)} (x' - x) \right\|.$$

2. Lemma 2. Consider the commuting family of nilpotent matrices $\{N_p : p \in \mathcal{P}\}$. For each $\delta > 0$, there is a constant $c_\delta > 0$ such that for all $v \in \mathbb{C}^n$,

$$c_\delta^{-1} e^{-\delta t} \|v\| \leq \left\| e^{\sum_{p\in\mathcal{P}} N_p \tau_p(t)} v \right\| \leq c_\delta e^{\delta t} \|v\|$$

   for all $t \geq 0$ with the active times τ_p.

3. Given a radius $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that

$$\cdots \leq \|\xi_\sigma(x',t) - \xi_\sigma(x,t)\| \leq c_\varepsilon e^{\varepsilon t} \max_{i=1,\ldots,n} e^{\sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t)} |x_i' - x_i|.$$

4. For the upper/lower bound, construct a (T,ε)-spanning/separated set using a grid. □

The Non-Switched Case

The formula for entropy yields the following well-known result [Bowen, 1971]:

Corollary 7

The topological entropy of the linear time-invariant (LTI) system

$$\dot{x} = Ax, \qquad x(0) \in K$$

equals the sum of the positive real parts of the eigenvalues of A, that is,

$$h(A) = \sum_{\lambda \in \mathrm{spec}(A)} \max\{\mathrm{Re}(\lambda),\, 0\}.$$

Proof.

1. The spectrum $\mathrm{spec}(A) = \{a^1, \ldots, a^n\}$.
2. The entropy

$$h(A) = \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \mathrm{Re}(a^i)t = \sum_{i=1}^{n} \max\{\mathrm{Re}(a^i),\, 0\}. \quad \square$$

More General Upper and Lower Bounds for Entropy

#	Formula/upper bounds	Sw	CoB
(1)	$= \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t)$	τ_p	Yes
(2)	$\leq \sum_{i=1}^{n} \max\left\{ \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\rho_p(t),\ 0 \right\}$	ρ_p	Yes
(3)	$\leq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} h(D_p)\rho_p(t)$	ρ_p	No
(4)	$\leq \sum_{p\in\mathcal{P}} h(D_p)\hat{\rho}_p$	$\hat{\rho}_p$	No
(5)	$\leq \max_{p\in\mathcal{P}} h(D_p)$	N/A	No
(6)	$\leq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} n\\|A_p\\|\rho_p(t)$	ρ_p	No
(7)	$\geq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(A_p)\rho_p(t)$	ρ_p	No

Numerical Example

Let $\mathcal{P} = \{1,\, 2\}$ and

$$D_1 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

	$(\hat{\rho}_1, \hat{\rho}_2)$	(1)	(2)	(3)	(4)	(5)	(6)	(7)
No switch	$(1, 0)$	2	2	2	2	3	4	1
Periodic switches	$(0.5, 0.5)$	2	2	2.5	2.5	3	5	2
Switches w/ set-points	$(0.9, 0.9)$	2.8	4.4	2.9	4.5	3	5.8	2.8

Conclusion

Contributions:

- Switched linear systems with pairwise commuting matrices.
- A change of basis under which each of the matrices can be decomposed into a diagonal part and a nilpotent part, and all the diagonal and nilpotent parts are pairwise commuting.
- A formula for the topological entropy, which only depends on the diagonal part.
- More general upper bounds for the entropy.

Future research:

- Reconcile the switching characterizations for entropy computation and for stability analysis and control design
    - Stability and stabilization: slow-switching conditions such as the average dwell-time
    - Entropy: the active time (rarely seen in the literature)

References I

R. L. Adler, A. G. Konheim, and M. H. McAndrew, "Topological entropy," *Transactions of the American Mathematical Society*, vol. 114, no. 2, pp. 309–319, 1965.

R. Bowen, "Entropy for group endomorphisms and homogeneous spaces," *Transactions of the American Mathematical Society*, vol. 153, pp. 401–414, 1971.

F. Colonius, "Minimal bit rates and entropy for exponential stabilization," *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 2988–3010, 2012.

F. Colonius and C. Kawan, "Invariance entropy for control systems," *SIAM Journal on Control and Optimization*, vol. 48, no. 3, pp. 1701–1721, 2009.

E. I. Dinaburg, "The relation between topological entropy and metric entropy," *Doklady Akademii Nauk SSSR*, vol. 190, no. 1, pp. 19–22, 1970, in Russian.

J. P. Hespanha, A. Ortega, and L. Vasudevan, "Towards the control of linear systems with minimum bit-rate," in *15th International Symposium on Mathematical Theory of Networks and Systems*, 2002, pp. 1–15.

J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer New York, 1972, vol. 9.

A. N. Kolmogorov, "A new metric invariant of transitive dynamical systems and automorphisms of Lebesgue spaces," *Doklady Akademii Nauk SSSR*, vol. 119, no. 5, pp. 861–864, 1958, in Russian.

D. Liberzon, "Finite data-rate feedback stabilization of switched and hybrid linear systems," *Automatica*, vol. 50, no. 2, pp. 409–420, 2014.

D. Liberzon and S. Mitra, "Entropy and minimal bit rates for state estimation and model detection," *IEEE Transactions on Automatic Control*, vol. 63, no. 10, pp. 3330–3344, 2018.

G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, no. 4, pp. 585–593, 2003.

References II

G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran, "Topological feedback entropy and nonlinear stabilization," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1585–1597, 2004.

K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2469–2471, 1994.

A. V. Savkin, "Analysis and synthesis of networked control systems: topological entropy, observability, robustness and optimal control," *Automatica*, vol. 42, no. 1, pp. 51–62, 2006.

H. Sibai and S. Mitra, "Optimal data rate for state estimation of switched nonlinear systems," in *20th International Conference on Hybrid Systems: Computation and Control*, 2017, pp. 71–80.

S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

G. Yang and D. Liberzon, "Feedback stabilization of a switched linear system with an unknown disturbance under data-rate constraints," *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 2107–2122, 2018.

G. Yang, A. J. Schmidt, and D. Liberzon, "On topological entropy of switched linear systems with diagonal, triangular, and general matrices," in *57th IEEE Conference on Decision and Control*, 2018, to appear.