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State Estimation for Asynchronously Switched Sampled-Data Systems

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***Abstract*— Asynchronously switched sampled-data systems can help model power systems and vehicles that evolve in continuous-time with switching behavior and discrete time measurements. We address the problem of jointly estimating a switching signal, with uncertainty in the exact switching times, as well as the continuous states of the system. We prove stability of the standard Kalman Filter under uncertainty in the switching times, with statistical bounds relating to the sampling period. We then propose a method for estimation of switching times as well as a method for efficient joint estimation of the state and switching signal inspired by the interacting multiple-model extended-Viterbi algorithm. We validate our algorithms in simulation for a power converter and a maneuvering vehicle.**

I. Introduction

Real-world systems are often best modeled in continuous time, for example using equations of motion, but with measurements taken at discrete instants [1]. Many systems also vary their behavior between discrete modes either by their construction or to simplify control [2]; for example, geared robot motion, power systems using switched circuits or sources, or an aircraft with several trim conditions including cruising and banked turning. In real-world systems we must also consider noise in our process and measurements, usually represented by random additive noise. A practical formulation for such systems is a stochastic sampled-data switched system [3], given by

$$\dot{x}(t) = f(\sigma(t), x(t), u(t)) + w(t)$$
$$y(t_k) = h(\sigma(t_k), x(t_k)) + v(t_k),$$

where $x(t)$ is the state, $u(t)$ is an input, $w(t)$ is a process noise, $y(t_k)$, usually denoted y_k, is a measured output subject to random measurement noise $v(t_k)$, usually denoted v_k, $\sigma(t)$ is a "switching signal" taking values in a finite set that tells us the active mode at time t, and t_k are discrete times of measurements indexed by k. The control of such systems is addressed in [4].

State estimation of discrete-time switched systems has attracted considerable attention, including work by Alessandri et al [5]. In these papers, the unknown switching signal is estimated using a Maximum-Likelihood method combined with either Kalman Filtering or Moving Horizon Estimation of the continuous states. In contrast, Interacting Multiple-Model (IMM) approaches to hybrid system state estimation have been suggested in [6] and [7]. Ho [8] augmented these methods using Viterbi algorithm concepts to obtain pseudo Maximum-A-Posteriori (MAP) solutions to the windowed estimation problem. In [9], a review of estimation methods for switched systems is provided.

In these prior works, it is always assumed that switches occur only at times that measurements are obtained, in other words the sampling times. There are papers that consider estimation of continuous-time switched systems like [10], [11], and [12]. In [13], the authors consider switches that occur at a constant offset from the measurement times. However we could not find prior works that consider the problem of fully asynchronous switches with sampled measurements.

In this paper we address state estimation when switches can occur at any time between measurement samples. In Section III we provide results on the convergence of Kalman Filter error in the setting where the switching signal is known at sampling times, but exact switching times are unknown. We build upon analysis first done by Anderson and Moore [14], and more recently extended by Zhang [15]. We provide bounds on the mean error and mean-squared error (MSE) of the estimates that can be useful in the context of control [16].

In Section IV we provide a method for simultaneously estimating the state $x(t)$ and switching signal $\sigma(t)$. This method is inspired by the IMM Extended Viterbi (IMM-EV) Approach [8]. In Section V, we show simulations that demonstrate our theoretical results and validate the performance of our algorithms.

II. Preliminaries

We consider a linear sampled-data output-error switched system with continuous-time dynamics,

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \tag{1}$$
$$y_k = Hx(t_k) + v_k, \tag{2}$$

for $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^\ell$, $v_k, y_k \in \mathbb{R}^m$, and $A(\sigma(t)) \in \{A(1), \ldots, A(L)\}$ a $n \times n$ matrix, $B(\sigma(t)) \in \{B(1), \ldots, B(L)\}$ a $n \times \ell$ matrix, with switching signal $\sigma(t) \in \{1, \ldots, L\}$. Our goal is to jointly estimate the switching signal and state at discrete periodically sampled timesteps $t_k = kT$, where T is the sampling period. We denote the state, input, and active mode at the discrete timesteps as $x_k = x(t_k)$, $u_k = u(t_k)$ and $\sigma_k = \sigma(t_k)$ respectively, as well as the active system matrices $A_k = A(\sigma(t_k))$ and $B_k = B(\sigma(t_k))$. We impose a dwell time $\tau_d > T$ so that switches occur at least τ_d apart from each other and at most once per sample. We can then parametrize the signal $\sigma(t)$ by the sequences $\{\sigma_k\}$ and $\{\bar{t}_k\}$, for each $t_k \in [0, t_{k+1} - t_k)$

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specifying the exact time at which a switch occurs within the interval $[t_k, t_{k+1})$, or an arbitrary value if no switch occurs.

III. Kalman Filter Convergence

We assume a zero order hold (ZOH) for the input, so that we have an exact discrete time update equation

$$x_{k+1} = F_k x_k + G_k u_k, \tag{3}$$

where

$$F_k = e^{A_{k\ \ 1}(T-\bar{t}_k)} e^{A_k \bar{t}_k}. \tag{4}$$

and

$$G_k = F_k \int_0^{\bar{t}_k} e^{-A_k \tau} B_k d\tau + \int_{\bar{t}_k}^{T} e^{A_{k\ \ 1}(T-\tau)} B_{k+1} d\tau. \tag{5}$$

A. Conditions for Observability

We consider the following definition of observability for a discrete-time linear time-varying system [15].

Definition 1 (Uniform Observability): The sequence (F_k, H) is uniformly observable i.e. there exist constants $h \in \mathbb{Z}_{>0}$ and $\rho_1 \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$

$$\rho_1 \|x\|^2 \leqslant x' \left(\sum_{i=k}^{k+h} \Phi'_{k+h,i} H' R^{-1} H \Phi_{k+h,i} \right) x$$

where $\Phi_{i,k} \coloneqq F_{i-1} \cdots F_{k+1} F_k$.

In many cases, uniform observability of time-varying systems like switched systems is difficult to verify for all possible switching signals [17]. By imposing a dwell time τ_d, uniform observability of each mode can generate uniform observability of the switched system.

Assumption 1 (Each mode observable): Suppose that each unswitched pair $\{(e^{A(1)T}, H), \ldots, (e^{A(L)T}, H)\}$ represents a uniformly observable system with constants $h_1, \ldots, h_L$, $\rho_1^1, \ldots, \rho_1^L$.

Lemma 1: Suppose that we have Assumption 1 and $\tau_d > \overline{h}T$, where $\overline{h} \coloneqq \max\{h_1, \ldots, h_L\}$, then the switched system in (3), (2) is uniformly observable for every admissible switching sequence with constants $h = 2\overline{h} - 1$ and $\rho_1 \geqslant \min\{\rho_1^1, \ldots, \rho_1^L\}$, that do not depend on the sequence.

Proof. Given that $\tau_d > \overline{h}T$, the system must spend greater than h_j timesteps in any mode j. In order to guarantee that the time window $[t_k, t_{k+h})$ contains at least h_j samples uninterrupted in a single mode j, then our window must be at least $h = 2\overline{h} - 1$ samples long. In any window of this length we must have,

$$\rho \|x\|^2 \leqslant x' \left(\sum_{i=k}^{k+2\overline{h}-1} \Phi'_{i,k} H' R^{-1} H \Phi_{i,k} \right) x$$

where $\rho = \min\{\rho_1^1, \ldots, \rho_1^L\}$. □

B. Errors in System Matrices

A Kalman Filter is the MAP state estimator of a discrete-time system. Kalman Filters compute state estimates $\hat{x}_k$ and their associated covariance matrices P_k at sample k. We assume that our initial condition is a random variable $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$ and that $v_k \sim$ iid $\mathcal{N}(0, R)$ for R an $m \times m$ symmetric positive-definite matrix. We compute the estimate at sample $k+1$ by combining y_{k+1} with a prediction $\hat{x}_{k+1|k}$ based on the previous estimate $\hat{x}_k$. These sources of information are combined through the Kalman gain matrix K_k which depends on the system matrices as well as the measurement noise variance R. When we do not know system matrices F_k and G_k exactly, due to uncertainty in switching times, but have estimates $\hat{F}_k$ and $\hat{G}_k$, then our output-error Kalman Filter update equations are of the form,

$$\hat{x}_{k+1|k} = \hat{F}_k \hat{x}_k + \hat{G}_k u \tag{6}$$
$$P_{k+1|k} = \hat{F}_k P_k \hat{F}'_k \tag{7}$$
$$K_k = (\hat{F}_k P_k \hat{F}'_k) H' (H (\hat{F}_k P_k \hat{F}'_k) H' + R)^{-1} \tag{8}$$
$$\hat{x}_{k+1} = (I - K_k H) \hat{x}_{k+1|k} + K_k y_{k+1} \tag{9}$$
$$P_{k+1} = (I - K_k H) P_{k+1|k}. \tag{10}$$

First we provide error bounds for our estimated system matrices assuming that we know the correct sequence $\{\sigma_k\}$ but not the exact switching times $\{\bar{t}_k\}$, instead using estimates $\{\hat{t}_k\}$ plugged into (4) and (5). In this situation we will bound the error of our state estimates using bounds on the error of the estimated state transition matrices due to switching time uncertainty.

Lemma 2 (Error in Estimation of System Matrices): For a transition between modes i and j, let the error in switching time estimation be denoted $\widetilde{t} \coloneqq \hat{t} - \bar{t}$, then the estimation error, $\widetilde{F} \coloneqq \hat{F} - F$ is bounded in norm as

$$\begin{aligned} \|\widetilde{F}\| &\leqslant |\widetilde{t}| \|A(j) - A(i)\| e^{(\|A(j)-A(i)\| + 3\|A(i)\| + \|A(j)\|)T} \\ &\leqslant T \|A(j) - A(i)\| e^{(\|A(j)-A(i)\| + 3\|A(i)\| + \|A(j)\|)T}. \end{aligned} \tag{11}$$

and the estimation error, $\widetilde{G} \coloneqq \hat{G} - G$ is bounded in norm as

$$\begin{aligned} \|\widetilde{G}\| \leqslant{} & \|\widetilde{F}\| e^{\|A(i)\|T} \|B(i)\| + |\widetilde{t}| e^{\|A(j)\|T} \\ & \cdot \left(e^{\|A(i)\|T} \|B(i)\| + e^{\|A(j)\|T} \|B(j)\| \right). \end{aligned} \tag{12}$$

Proof. Call $F_j = e^{A(j)(T-\bar{t})}$, $F_i = e^{A(i)\bar{t}}$, $E_j \coloneqq e^{-A(j)\widetilde{t}}$ and $E_i \coloneqq e^{A(i)\widetilde{t}}$. We then have that $\hat{F} = \hat{F}_j \hat{F}_i = F_j E_j E_i F_i$, so $\widetilde{F} = F_j (E_j E_i - I) F_i$, then

$$\begin{aligned} \|\widetilde{F}\| &\leqslant \|E_j E_i - I\| \|F_j\| \|F_i\| \\ &\leqslant \|(E_j - E_i^{-1}) E_i\| e^{(\|A(j)\| + \|A(i)\|)T} \\ &\leqslant \|(e^{-A(j)\widetilde{t}} - e^{-A(i)\widetilde{t}}) e^{A(i)\widetilde{t}}\| e^{(\|A(j)\| + \|A(i)\|)T}. \end{aligned}$$

Then using the fact that $\|e^{X+Y} - e^X\| \leqslant \|Y\| e^{\|X\| + \|Y\|}$ [18] where $Y = -A(j)\widetilde{t} - A(j)\widetilde{t}$ and $X = -A(i)\widetilde{t}$, we obtain (11).

Our error in G, after some manipulation, can be written as

$$\widetilde{F}\int_0^{\hat{t}} e^{-A(i)\tau}B(i)d\tau + e^{A(j)(T-\bar{t})}\int_0^{\widetilde{t}} e^{-A(i)\tau}B(i)d\tau$$
$$- e^{A(j)(T-\bar{t})}\int_0^{\widetilde{t}} e^{A(j)\tau}B(j)d\tau$$

from which we obtain

$$\begin{aligned}\|\widetilde{G}\| \leqslant \|\widetilde{F}\| \int_0^{\hat{t}} e^{\|A(i)\|\tau}\|B(i)\|d\tau \\ + e^{\|A(j)\|T}\int_0^{\|\widetilde{t}\|}\big(e^{\|A(i)\|\tau}\|B(i)\| \\ + e^{\|A(j)\|\tau}\|B(j)\|\big)d\tau\end{aligned}$$

which gives us (12) after computing integrals. □

The bounds in (11) and (12) guarantee that the errors in $\hat{F}$ and $\hat{G}$ go to zero as $\widetilde{t}$, the error in $\hat{t}$, goes to zero, which happens when our sampling period T goes to zero. These bounds also improve if the $A(i)$'s are approximately equal.

C. Bounds on Estimation Errors

To bound the estimation mean and mean-squared error of our filter, we denote the filter error by $e_k := \hat{x} - x$, and the prediction error by $z_{k+1} := x_{k+1} - \hat{F}\hat{x}_k - \hat{G}_k u_k$. With switching time uncertainty, the error propagates as

$$e_{k+1} = (I - K_k H)z_{k+1} + K_k v_{k+1} \tag{13}$$

where in a sampling period in which no switch occurs,

$$z_{k+1} = \hat{F}_k e_k \tag{14}$$

and in a period where a switch occurs,

$$z_{k+1} = \hat{F}_k e_k + \widetilde{F}_k x_k + \widetilde{G}_k u_k. \tag{15}$$

We define the mean squared errors $\Sigma_k := \mathbb{E}[e_k e_k']$ and $\Omega_k := \mathbb{E}[z_k z_k']$. These update as

$$\Sigma_{k+1} = (I - K_k H)\Omega_{k+1}(I - K_k H)' + K_k R K_k'. \tag{16}$$

where when no switch occurs,

$$\Omega_{k+1} = \hat{F}_k \Sigma_k \hat{F}_k' \tag{17}$$

and when a switch occurs,

$$\begin{aligned}\Omega_{k+1} = \hat{F}_k\Sigma_k\hat{F}_k' + \hat{F}_k\mathbb{E}[e_k x_k']\widetilde{F}_k' + \widetilde{F}_k\mathbb{E}[x_k e_k']\hat{F}_k \\ + \widetilde{F}_k\mathbb{E}\left[x_k x_k'\right]\widetilde{F}_k' + \hat{F}_k\mathbb{E}[e_k]u_k'\widetilde{G}_k \\ + \widetilde{F}_k\mathbb{E}\left[x_k\right]u_k'\widetilde{G}_k' + \widetilde{G}_k u_k\mathbb{E}\left[x_k\right]'\widetilde{F}_k' \\ + \widetilde{G}_k u_k\mathbb{E}[e_k]'\hat{F}_k' + \widetilde{G}_k u_k u_k'\widetilde{G}_k'.\end{aligned} \tag{18}$$

We will need the following:

Assumption 2: Suppose that $P_{k|k-1}$ is positive-definite and bounded above for all k. The upper bound is shown in [15] while the lower bound is usually ensured by an additional term in the filter as shown later. Let $\overline{\lambda}$ denote the maximum, and $\underline{\lambda}$ the minimum eigenvalue that $P_{k|k-1}^{-1}$ can have.

Fact 1 (Observability of error dynamics): In [15] it is shown that if the sequence $(\hat{F}_k, H)$ uniformly observable then the sequence $(\hat{F}_k(I - K_{k-1}H), H)$ is also uniformly observable, i.e. there exists $\rho_3 \in \mathbb{R}_{>0}$ such that for the same h as in Definition 1,

$$\rho_3\|e\|^2 \leqslant e'\left(\sum_{i=k}^{k+h}\bar{\Phi}_{i,k}' H' R^{-1} H \bar{\Phi}_{i,k}\right)e$$

for all e, where $\bar{\Phi}_{i,k} := F_{i-1}(I - K_{i-2}H)\cdots F_k(I - K_{k-1}H)$.

We now present a theorem bounding the expected prediction error and mean-squared prediction error.

Theorem 1 (Bounds on prediction error): Given Assumptions 1 and 2, and suppose $\mathbb{E}\left[x_k' x_k\right] < \gamma^2$, and $\|u_k\| < \delta$ for all k, let

$$d := \frac{\alpha_3}{\rho_3}\overline{\lambda}(\gamma^2\|\widetilde{F}_k\|^2 + 2\gamma\delta\|\widetilde{F}_k\|\|\widetilde{G}_k\| + \delta^2\|\widetilde{G}_k\|^2)$$

where $\alpha_3 = 1 + \alpha_1/\alpha_2$, $\alpha_1 > 0$ the largest possible eigenvalue of $H'P_{k|k-1}H$ for all k, and $\alpha_2 > 0$ the smallest eigenvalue of R. Then there exist constants $\beta > 0$, $\xi > 0$, and function $c(\cdot)$ given by

$$c(a) := \frac{\overline{\lambda}}{\underline{\lambda}}\left(a + \beta\sqrt{a} + \xi\right)$$

for $a \in \mathbb{R}_{>0}$, such that for any $i \in \mathbb{Z}_{>0}$,

$$\|\mathbb{E}\left[z_{k+i}\right]\|^2 \leqslant \max\left\{c\left(\|\mathbb{E}\left[z_k\right]\|^2\right), c\left(c(d)\right)\right\} \tag{19}$$

Furthermore, there exist constants $\omega_h > 0$ and $\omega_{h-1} > 0$, such that the prediction MSE is bounded for all times $k + j$, $j \in \mathbb{Z}_{>0}$ as

$$\begin{aligned}&\mathrm{tr}(\Omega_{k+j}) \\ &\leqslant \max\left\{\frac{\overline{\lambda}}{\underline{\lambda}}\left(\mathrm{tr}(\Omega_k) + 2\omega_{h-1}\right), \frac{\overline{\lambda}}{\underline{\lambda}}\left(\frac{2\overline{\lambda}\sigma_3}{\rho_3}\omega_h + 2\omega_h\right)\right\}\end{aligned} \tag{20}$$

A proof is provided in the Appendix.

Remark 1 (Estimation Error Bounds): Given the bounds in Theorem 1, we can also bound $\mathbb{E}\left[e_k\right]$ and Σ_k for arbitrary k using

$$\|\mathbb{E}\left[e_k\right]\| \leqslant \|I - K_{k-1}H\|\|\mathbb{E}\left[z_k\right]\| \tag{21}$$

and

$$\mathrm{tr}(\Sigma_k) \leqslant \|I - K_{k-1}H\|^2\mathrm{tr}(\Omega_k) + \|K_{k-1}\|^2\mathrm{tr}(R) \tag{22}$$

which follow from (13) and (16) respectively.

This theorem and remark state that the dwell time condition ensures that intermittent model uncertainties due to switching do not lead to unbounded growth in our state estimation errors. We will now present an algorithm that allows us to exploit this property.

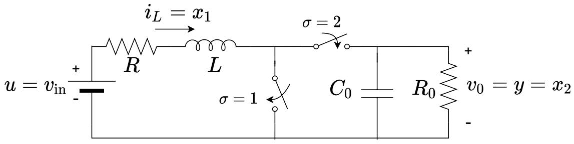

Fig. 1. Boost Converter Circuit

IV. Joint Estimation of State and Switching

In Theorem 1, the error in the state estimates is driven by the switching time errors appearing in Lemma 2. We will augment the IMM extended viterbi [8] with the following maximum-likelihood approach to estimating the switching time within a single sample interval,

$$J_k(\tau) = p\big(y_{k+1}|x_{k+1} = \hat{x}_{k+1|k,\bar{t}_k=\tau}\big)$$

Where $\hat{x}_{k+1|k,\bar{t}=\tau}$ is computed either using (4) (5), or symbolic discretizations of the initial and final modes. We can search for the optimum of this cost by gridding the sample period $[t_k, t_{k+1})$ with g points $\{\tau_i\}_1^g$, for example where $\tau_i \coloneqq \frac{iT}{g} - \frac{T}{2g}$. We can then compute

$$\hat{t}_k = \arg\max_{\tau_i} J_k(\tau_i). \tag{23}$$

There are many gridding schemes that could be equivalently used here. We can now state Algorithm 1, a heuristic method which builds on the IMM-EV1 Kalman Filter by including our gridded switching time estimation. To ensure that $\overline{\lambda}$ exists in Assumption 2, we add ϵI to each $P_{k|k-1}$. This is common practice in the design of practical KFs [19] and does not affect our analysis, as only bounds on the terms of the sequence $\{P_k\}$ are needed, which are ensured by the observability condition [8][14].

Algorithm 1 IMM-EV1 Kalman Filter

filter bank $\{(\hat{x}_k^1, P_k^1), \dots, (\hat{x}_k^L, P_k^L)\}$
mode probabilities $a_k^1, \dots, a_k^L$
for i from 1 to L **do**
  for j from 1 to L **do**
    compute $\hat{t}_k^{ij}$ for switch from i to j using (23)
    let $b_{ij} = J_k(\hat{t}_k^{ij})$
  end for
  $\hat{j} = \max_j b_{ij}$
  compute $\hat{x}_{k+1}^i$, and P_{k+1}^i from $\hat{x}_k^{\hat{j}}$, $P_k^{\hat{j}}$, and $\hat{t}_k^{i\hat{j}}$,
  $a_{k+1}^i = b_{i\hat{j}} a_k^{\hat{j}}$
end for
normalize a_{k+1}^i's

V. Simulations

In this sections we provide simulations to validate our theory and joint estimation algorithms.

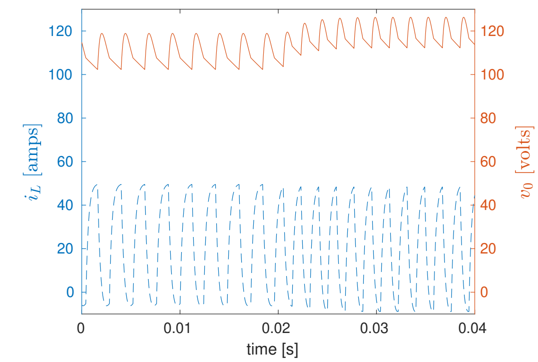

Fig. 2. Example state evolution for Boost Converter starting with switches every 1.2 ms then increasing to every 0.9 ms at 0.02 seconds, blue dashed line indicates i_L and red solid line indicates v_0.

T/g (ms)	0.5	0.25	0.167	0.125	0.1
RMSE(i_L) (amps)	13.5	10.7	9.40	9.17	9.09
RMSE(v_0) (volts)	2.74	2.15	2.00	1.96	1.89

TABLE I

Effect of increasingly precise gridded switching time estimation on Kalman Filter RMSE, 100 trials

A. Boost Converter

A Boost Converter is a popular switching power converter for stepping up a DC voltage without transformers or amplifiers. This is necessary when a high-power source is not available to perform amplification. A model for a realistic boost converter is provided in [20]. We have dynamics as given in (1) where

$$A(1) = \begin{bmatrix} -R_1/L_1 & 0 \\ 0 & -1/R_0C_0 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -R_1/L_1 & -1/L_1 \\ 1/C_0 & -1/R_0C_0 \end{bmatrix},$$
$$B(1) = B(2) = [1/L_1 \;\; 0]'$$

Where $x = [i_L \; v_0]'$ and $u = v_{in}$. We additionally choose $y = v_0$, or in other words $H = [0 \; 1]$. We use the values $R_1 = 2\Omega$, $L_1 = 500\mu\text{H}$, $R_0 = 50\Omega$, $C_0 = 470\mu\text{F}$, and $v_{in} =$ 100 volts from [20]. When switches occur every millisecond, these parameters result in an output voltage around 110 volts. Figure 2 shows the result of simulating this system.

We simulated 10 seconds of operation with switch frequencies ranging from 1.2 to 0.9 ms/switch, and output voltages ranging between 100 and 120 volts, with measurement noise corresponding to $R = 5$ volts^2. Table I shows how effective the gridded estimation in (23) is when sampling at 0.35 ms for different values of g. As expected more precision in the switching time interval leads to more accuracy in the Kalman Filter estimates.

B. Vehicle Maneuver Tracking

A model of a continuous-time switched system representing a vehicle moving in two dimensions with $x =$

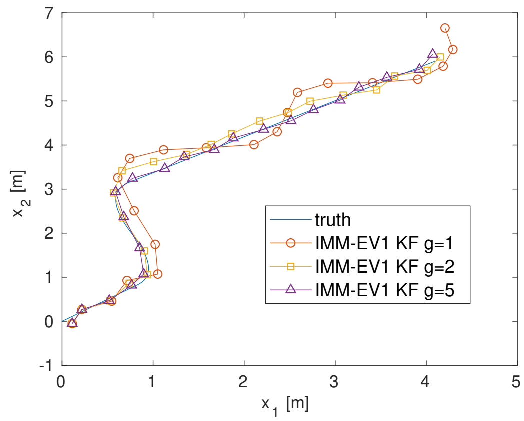

Fig. 3. Single trial of vehicle true and estimated trajectories for $g = 1, 2, 5$

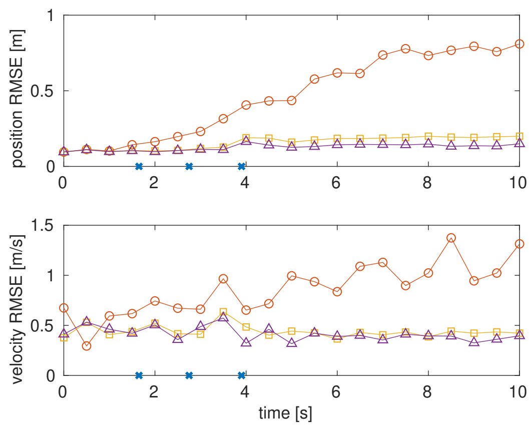

Fig. 4. Vehicle Position and Velocity Estimation RMSE over 500 MC Trials, filtering with $g = 2$ shown with red circles, $g = 5$ with yellow squares, and $g = 10$ with purple triangles. Blue x's mark the switching times.

$[x_1\ \dot{x}_1\ x_2\ \dot{x}_2]'$ is given by,

$$A(1) = A(2) = A(3) = I_2 \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} B(1) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}' \\ B(2) &= \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}' \\ B(3) &= \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}' \end{aligned}$$

where $\otimes$ denotes the Kronecker product, with $\tau_d > 1$ second and measurements sampled every 0.5 seconds. This double-integrator system corresponds to the discrete time switched systems used in [8], among others. Its discretization with ZOH over timestep T is given by

$$F(1) = F(2) = F(3) = I_2 \otimes \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{24}$$

$$\begin{aligned} G(1) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}' \\ G(2) &= \begin{bmatrix} -\frac{T^2}{2} & -T & \frac{T^2}{2} & T \end{bmatrix}' \\ G(3) &= \begin{bmatrix} \frac{T^2}{2} & T & -\frac{T^2}{2} & -T \end{bmatrix}'. \end{aligned}$$

We consider a single trajectory over 10 seconds, with $u(t) = 1$, starting in mode 1, swiching to mode 2 at 1.65 seconds, to mode 3 at 2.75 seconds, and back to mode 1 at 3.9 seconds. The resulting trajectory is shown in Figure 3 along with estimate trajectories computed using Algorithm 1 where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}.$$

We compute the RMSE over 100 monte carlo trials, and the results for varying divisions, g, of our sampling time are show in Figure 4.

VI. Conclusions

We examined the case of intermittent uncertainty in switched system dynamics due to unknown switching times. We developed bounds on the Kalman Filter error under dwell-time constraints which give us intuition for implementing filters like the IMM-EV1 KF. Validating our theory, simulations of a boost converter and a maneuvering vehicle showed improvement in the accuracy of filtering algorithms when we improved the precision of switching time estimates.

An immediate extension is to consider requirements on control algorithms to satisfy the assumptions in Theorem 1, as well as relaxing the ZOH and bounding loss of optimality. It would be interesting to extend these results to nonlinear problems, for which our analysis could be applied to linearized error dynamics.

Appendix
Proof of Theorem 1

From (13)-(15) we get that

$$\mathbb{E}[z_{k+1}] = \hat{F}_k(I - K_{k-1}H)\mathbb{E}[z_k]$$

when no switch occurs between samples k and $k+1$, and

$$\mathbb{E}[z_{k+1}] = \hat{F}_k(I - K_{k-1}H)\mathbb{E}[z_k] + \widetilde{F}_k\mathbb{E}[x_k] + \widetilde{G}_k u_k$$

when a switch occurs. We use the Lyapunov function

$$V_k \coloneqq \mathbb{E}[z_k]' P_{k|k-1}^{-1}\mathbb{E}[z_k]$$

which from Assumption 2 is positive definite and upper bounded. We have that

$$V_{k+1} - V_k = -\mathbb{E}[z_k]' H' S_k^{-1} H \mathbb{E}[z_k] \tag{25}$$

when no switch occurs, where $S_k \coloneqq H P_{k+1|k} H' + R$. When a switch occurs,

$$\begin{aligned} V_{k+1} - V_k = &-\mathbb{E}[z_k]' H' S_k^{-1} H \mathbb{E}[z_k] \\ &+ 2q' P_{k+1|k}^{-1}\Lambda_k \mathbb{E}[z_k] + q' P_{k|k-1}^{-1} q. \end{aligned} \tag{26}$$

where $\Lambda_k \coloneqq \hat{F}_k(I - K_k H)$ and $q \coloneqq \widetilde{F}_k\mathbb{E}[x_k] + \widetilde{G}_k u_k$. It is derived in [15] that,

$$\mathbb{E}[z_k]' H' S_k^{-1} H \mathbb{E}[z_k] \leqslant -\frac{1}{\alpha_3}\mathbb{E}[z_k]' H' R^{-1} H \mathbb{E}[z_k] \tag{27}$$

for $\alpha_3 > 0$ defined in our theorem. For a switch occurring between times k and $k+1$ but no switches in the interval

$$\exists \pi > 0 \text{ s.t. } \pi \begin{bmatrix} -I & 0 \\ 0 & d \end{bmatrix} - \begin{bmatrix} -\frac{\rho_3}{\alpha_3} I & \Lambda_k'(P_{k|k-1}^{-1})'q \\ q'P_{k|k-1}^{-1}\Lambda_k & q'P_{k|k-1}^{-1}(\widetilde{F}_k x_k + \widetilde{G}_k u_k) \end{bmatrix} > 0 \tag{31}$$

$$\exists \pi > 0 \text{ s.t. } \frac{\rho_3}{\alpha_3} - \pi > 0 \text{ and } \pi d - q'P_{k|k-1}^{-1}q - (\frac{\rho_3}{\alpha_3} - \pi) q'P_{k|k-1}^{-1}\Lambda_k\Lambda_k'(P_{k|k-1}^{-1})'q > 0 \tag{32}$$

$\{k+1, \ldots, k+h\}$ we then know that $V_{k+h} - V_k$ is bounded above by

$$-\frac{1}{\alpha_3}\mathbb{E}[z_k]' \left(\sum_{i=k}^{k+h} \bar{\Phi}_{i,k}' H' R^{-1} H \bar{\Phi}_{i,k} \right) \mathbb{E}[z_k] + 2q'P_{k+1|k}^{-1}\Lambda_k \mathbb{E}[z_k] + q'P_{k|k-1}^{-1}q.$$

From uniform observability and Fact 1 we have

$$V_{k+h} - V_k \leqslant -\frac{\rho_3}{\alpha_3}\|\mathbb{E}[z_k]\|^2 + 2q'P_{k+1|k}^{-1}\Lambda_k\mathbb{E}[z_k] + q'P_{k|k-1}^{-1}q. \tag{28}$$

In other words, we now know that

$$\bar{\Phi}_{k+h,k}' P_{k+h|k+h-1}^{-1} \bar{\Phi}_{k+h,k} - P_{k|k-1}^{-1} \leqslant -\frac{\rho_3}{\alpha_3} I. \tag{29}$$

We want to show that for a switch occurring between samples k and $k+1$, the expected prediction error at sample $k+h$ satisfies,

$$\begin{cases} V_{k+h} < V_k & \text{if } \|\mathbb{E}[z(k)]\|^2 > d \\ \|\mathbb{E}[z(k+h)]\|^2 < c(d) & \text{if } \|\mathbb{E}[z(k)]\|^2 \leqslant d \end{cases}$$

for some constant $d > 0$, and positive continuous function $c(\cdot)$. We proceed by considering the two cases:

1) Suppose $\|\mathbb{E}[z_k]\|^2 > d$. We want to show that

$$\|\mathbb{E}[z_k]\|^2 > d \Rightarrow V_{k+h} - V_k < 0 \tag{30}$$

Applying S-procedure [21] to (28), we know that (30) is true if and only if (31) is true. By Schur complement, this is equivalent to (32). If we choose $\pi = \frac{\rho_3}{\alpha_3} - \varepsilon$ for some small enough $\varepsilon > 0$, such that if

$$d > \frac{\alpha_3}{\rho_3}\overline{\lambda}(\gamma^2\|\widetilde{F}_k\|^2 + \gamma\delta\|\widetilde{F}_k\|\|\widetilde{G}_k\| + \delta^2\|\widetilde{G}_k\|^2)$$

then we satisfy the conditions in (32) and therefore show that the Lyapunov function decreases before the next switch occurs. Then the maximum value attained by $\|\mathbb{E}[z_{k+i}]\|^2$ for $i > 0$ satisfies

$$\|\mathbb{E}[z_{k+i}]\|^2 \leqslant \frac{\overline{\lambda}}{\underline{\lambda}}\big(\|\mathbb{E}[z_k]\|^2 + \beta\|\mathbb{E}[z_k]\| + \xi\big) \tag{33}$$

where

$$\beta \coloneqq 2\|\Lambda\|(\gamma\|\widetilde{F}_k\| + \delta\|\widetilde{G}_k\|) \tag{34}$$

$$\xi \coloneqq \gamma^2\|\widetilde{F}_k\|^2 + \gamma\delta\|\widetilde{F}_k\|\|\widetilde{G}_k\| + \delta^2\|\widetilde{G}_k\|^2 \tag{35}$$

2) Suppose $\|\mathbb{E}[z_k]\|^2 \leqslant d$. Then by substituting d into (26) we get

$$V_{k+1} \leqslant (\overline{\lambda} - \lambda_{\min}(\|H'S_k^{-1}H\|))d + \overline{\lambda}\beta\sqrt{d} + \overline{\lambda}\xi. \tag{36}$$

Since we showed that the Lyapunov function is non-increasing over a timestep with no switch and must decrease over h or more timesteps with no switch, then (36) gives us an upper bound on V_i for $k < i < k+j$ where $k+j$ is the sample where the next switch occurs. Then

$$\|\mathbb{E}[z_{k+j}]\|^2 \leqslant c(d) \coloneqq \frac{\overline{\lambda}}{\underline{\lambda}}\big(d + \beta\sqrt{d} + \xi\big) \tag{37}$$

which is a bound greater than d. If $\|\mathbb{E}[z_{k+j}]\|^2 > d$ then applying (33) to (37) tells us the maximum value attained by $\|\mathbb{E}[z_{k+i}]\|^2$ for $i > 0$ must satisfy

$$\|\mathbb{E}[z_{k+i}]\|^2 \leqslant c(c(d)) \tag{38}$$

(33) and (38) produce (19).

To prove (20) we will use the following Lyapunov function,

$$W_k \coloneqq \text{tr}(P_{k|k-1}^{-1} \cdot \Omega_k)$$

and proceed by similar analysis as with the expected error. We will use the following Ruhe trace inequality [22, Fact 5.12.4, p. 333]:

Fact 2: For positive semi-definite Hermitian matrices A and B with eigenvalues ordered largest to smallest, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$ respectively, the following holds

$$\sum_{i=1}^{n} a_{n-i+1}b_i \leqslant \text{tr}(AB) \leqslant \sum_{i=1}^{n} a_i b_i \tag{39}$$

which gives us

$$\underline{\lambda}\text{tr}(\Omega_k) \leqslant \text{tr}(P_{k|k-1}^{-1}\Omega_k) \leqslant \overline{\lambda}\text{tr}(\Omega_k) \tag{40}$$

where $\overline{\lambda}$ and $\underline{\lambda}$ are the maximum and minimum eigenvalues respectively attainable by $P_{k|k-1}^{-1}$ which are given by Assumption 2. We can bound the update of our Lyapunov function, $W_{k+1} - W_k$, over the step after a switch using (16)-(18), (39), (40), and the fact that $2\mathbb{E}[e'x] \leqslant \varepsilon\mathbb{E}[e'e] + \frac{1}{\varepsilon}\mathbb{E}[x'x]$ for arbitrary $\varepsilon > 0$ as

$$W_{k+1} - W_k \leqslant \text{tr}\Big(\big(\bar{\Phi}_{k+1,k}' P_{k+1|k}^{-1} \bar{\Phi}_{k+1,k} - P_{k|k-1}^{-1}\big)\Omega_k\Big) + \overline{\lambda}\varepsilon\|\hat{F}_k\|\|\widetilde{F}_k\|\text{tr}(\Sigma_k) + \overline{\lambda}T_k \tag{41}$$

where

$$\begin{aligned} T_{k+1} = {} & (\hat{F}_k\mathbb{E}[e_k] + \widetilde{F}_k\mathbb{E}[x_k])'\widetilde{G}_k u_k \\ & + u_k'\widetilde{G}_k'(\hat{F}_k\mathbb{E}[e_k] + \widetilde{F}_k\mathbb{E}[x_k]) \\ & + \frac{1}{\varepsilon}\widetilde{F}_k^2\mathbb{E}[x_k'x_k] + \widetilde{G}_k^2\|u_k\|^2 \\ & + \hat{F}_k K_{k-1} R K_{k-1}' \hat{F}_k' \end{aligned}$$

with a switch. The Lyapunov function change over h steps, $W_{k+h} - W_k$, is then bounded by

$$\mathrm{tr}\left(\left(\bar{\Phi}'_{k+h,k} P^{-1}_{k+h|k+h-1} \bar{\Phi}_{k+h,k} - P^{-1}_{k|k-1} + \varepsilon\eta I\right)\Omega_k\right) + \mathrm{tr}\left(P^{-1}_{k+h|k+h-1} \sum_{i=0}^{h-1} \bar{\Phi}'_{k+i|k} T_{k+h-1-i} \bar{\Phi}_{k+i|k}\right)$$

where $\eta := \overline{\lambda}\|\hat{F}_k\|\|\widetilde{F}_k\|\|I - K_{k-1}H\|$ and $T_i = \|\hat{F}_i K_{i-1} R K'_{i-1} \hat{F}'_i\|$ when $i \neq k$. From (40) and (29), we get

$$W_{k+h} - W_k \leqslant \left(\frac{-\rho_3}{\alpha_3} + \varepsilon\eta\right)\mathrm{tr}(\Omega_k) + \overline{\lambda}\mathrm{tr}\left(\sum_{i=0}^{h-1} \bar{\Phi}'_{k+h|k+i} T_{k+i} \bar{\Phi}_{k+h|k+i}\right) \tag{42}$$

We choose $\varepsilon = \rho_3/(2\alpha_3\eta)$, which also affects the value of T_k. Therefore we see that over any h steps, if the MSE at time k satisfies

$$\mathrm{tr}(\Omega_k) > \frac{2\overline{\lambda}\alpha_3}{\rho_3}\mathrm{tr}\left(\sum_{i=0}^{h-1} \bar{\Phi}'_{k+h|k+i} T_{k+i} \bar{\Phi}_{k+h|k+i}\right) \tag{43}$$

then $W_{k+h} - W_k \leqslant 0$. We note that it might be possible to achieve a better bound with different choice of ε. We must then consider the fact that unlike for the expected error in (27), the Lyapunov function W_i can now increase even in non-switch intervals due to the T_k terms. We will again deal with this by splitting into two cases. First let ω_j be defined as the upper bound derived from our upper bounds on F_k, K_k, etc., as well as bounds on $\widetilde{F}$ and $\widetilde{G}$ from Lemma 2, and bound on $\|\mathbb{E}[z_k]\|$ in (19), of the quantity

$$\mathrm{tr}\left(\sum_{i=0}^{j-1} \bar{\Phi}'_{k+j|k+i} T_{k+i} \bar{\Phi}_{k+j|k+i}\right) \leqslant \omega_j,$$

for any k. We know that for any $j > 0$,

$$W_{k+j} - W_k \leqslant \overline{\lambda}\omega_j$$

Let us consider the two cases:

1) Suppose $\mathrm{tr}(\Omega_k) > \frac{2\overline{\lambda}\alpha_3}{\rho_3}\omega_h$. Then (41) and (42) tell us that $W_{k+h} < W_k$ and the maximum value between k and $k+h$ is bounded as

$$\mathrm{tr}(\Omega_{k+j}) \leqslant \frac{\overline{\lambda}}{\underline{\lambda}}\left(\mathrm{tr}(\Omega_k) + \omega_{h-1}\right) \text{ for } j \in \{k, \ldots, k+h\}, \tag{44}$$

which is also the maximum value attained until some $\mathrm{tr}(\Omega_{k+j}) \leqslant \frac{\overline{\lambda}\alpha_3}{\rho_3}\omega_h$, since the value cannot increase over h steps otherwise. This brings us to our next case:

2) Suppose $\mathrm{tr}(\Omega_k) \leqslant \frac{2\overline{\lambda}\alpha_3}{\rho_3}\omega_h$. Now the maximum value that $\mathrm{tr}(\Omega_{k+1})$ could attain is

$$\mathrm{tr}(\Omega_{k+1}) \leqslant \frac{\overline{\lambda}}{\underline{\lambda}}\left(\frac{2\overline{\lambda}\alpha_3}{\rho_3}\omega_h + \omega_1\right)$$

If we achieved the maximum then $\mathrm{tr}(\Omega_{k+1}) > \frac{2\overline{\lambda}\alpha_3}{\rho_3}\omega_h$, so $W_{k+h+1} \leqslant W_{k+1}$ and therefore the maximum value of $\mathrm{tr}(\Omega_{k+j})$ for all $j > 0$ is bounded as

$$\mathrm{tr}(\Omega_{k+j}) \leqslant \frac{\overline{\lambda}}{\underline{\lambda}}\left(\frac{2\overline{\lambda}\alpha_3}{\rho_3}\omega_h + \omega_h\right) \quad j \in \mathbb{Z}_{>0} \tag{45}$$

with (44) and (45) combine to prove (20), with an additional ω_{h-1} or ω_h added to each to account for the case of starting in non-switch timestep. □

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