Stabilization of Networked Control Systems Under Clock Offsets and Quantization

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***Abstract*—This paper studies the impact of clock mismatches and quantization on networked control systems. We consider a scenario where the plant's state is measured by a sensor that communicates with the controller through a network. Variable communication delays and clock jitter do not permit a perfect synchronization between the clocks of the sensor and controller. We investigate limitations on the clock offset tolerable for stabilization of the feedback system. For a process with a scalar-valued state, we show that there exists a tight bound on the offset above which the closed-loop system cannot be stabilized with any causal controllers. For higher dimensional plants, if the plant has two distinct poles, then the effect of clock mismatches can be canceled with a finite number of measurements, and hence there is no fundamental limitation. We also consider the case where the measurements are subject to quantization in addition to clock mismatches. For first-order plants, we present necessary conditions and sufficient conditions for stabilizability, which show that a larger clock offset requires a finer quantization.**

***Index Terms*—Clock offsets, continuous time systems, networked control systems, stabilizability, time-varying sampling periods, quantization.**

I. Introduction

The components of networked control systems are often spatially distributed and hence need to communicate through digital networks. A significant challenge in such systems is that the processors in the components may not share a common notion of time, because their local clocks are not properly synchronized; see survey papers [1], [2]. Quantization is another significant challenge to the design of networked control systems, as surveyed in [3] and [4] and motivated the study of bit-rate constraints in channels [5] and limited capabilities of sensors/actuators [6]. In this paper, we consider clock mismatches and quantization simultaneously and study the impact of such imperfections on the stability of feedback control systems.

With a growing demand for digital communication, there has been a vast amount of research providing clock synchronization methodologies over networks; see, e.g., [7]–[9] and the survey papers [10], [11]. Moreover, the global positioning system (GPS) has allowed us to have access to a global clock that has been used in many practical systems. However, we cannot attain perfect synchronization in most practical situations. In fact, the work in [12] has shown that synchronization of clocks with unknown skews and offsets is impossible in the presence of unknown communication delays. Furthermore, recent works [13], [14] have pointed out that GPS signals can be spoofed and hence are vulnerable against attacks.

In this paper, we consider feedback systems where the sensor measuring the plant state and the controller do not share a common clock. This asynchronism results in measurement that can cause fundamental limitations in our ability to stabilize a system through feedback. The clock mismatches cause uncertainty on sampling instants, which has been recognized as an important topic in control as there are many related works [15]–[20]. Precision of sensor clocks is of interest also from practical viewpoints. In [21], an algorithm exploiting time-stamps is presented which can compensate delays in a distributed control system. In mechatronic systems where sampling periods are fast (less than 1 ms), the effect of clock mismatches becomes relatively severe and hence compensation methodologies would be employed [22].

We consider the situation in which the clock offset between the sensor and the controller is known to be bounded, but its precise value is unknown and potentially time varying. Our objective is to clarify limitations on the tolerable clock offsets for stability. We first consider the case in which the communication channel has unlimited bandwidth and thus can convey real-valued state measurements. In such configuration, for a process with a scalar-valued state, we derive a necessary and sufficient condition for stabilizability, which gives a tight upper bound on the clock offset beyond which stabilization is impossible. In contrast, for a process with a vector-valued state, there exists no fundamental limitation on the offset if the plant has at least two distinct modes.

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We also consider the scenario where the sensor measurements must be quantized to discrete values. We consider quantizers that are static piecewise constant functions, which map the observed state to a discrete set of values. We focus our attention on two types of quantizers that have been widely considered in the literature: Logarithmic quantizers [23]–[25] and uniform quantizers [26]–[28]. It has been shown that, by itself, quantization imposes fundamental limitations on stabilizability [23], [28], [29]. Here, we introduce the additional complexity of clock mismatch and derive necessary conditions and sufficient conditions on coarseness of the quantizer and the maximum clock offset for stabilizability. The conditions provide explicit relationships between the tolerable clock offsets and the required quantization levels for stabilizability.

Stabilization of systems with uncertain sampling intervals have been studied in various ways. It has been regarded as systems with input delay [15], [16] and uncertain impulsive systems [17], [18]. Polytopic overapproximation [19] and the gridding approach [20] are also notable as an instrument to deal with such uncertainty. These papers restrict their attention to linear time-invariant (LTI) controllers and sufficient conditions are provided. In a recent work [30], we employed a time-stamp aware estimator and investigated the stabilization problem with *static* clock offsets. The problem is modeled as simultaneous stabilization for parametric uncertainty, and a sufficient condition for the existence of stabilizing LTI controllers is given. Now we consider *time-varying* clock offsets and the controller's class which includes general causal systems. We present necessity results for stabilizability rather than only sufficiency ones. These results thus characterize fundamental requirements on the accuracy of local clocks for stabilizability.

For the analysis of stabilizability, we compute the state estimation set and investigate under which conditions it shrinks over time. By computing the *tight* estimation set and its decay rate, we are able to show necessity results. It is worth noticing that in the literature listed above contraction of Lyapunov functions or Lyapunov–Krasovskii functionals are studied to verify stability. We also note recent works [31], [32] that fall in this context. While in our approach the decay rate is determined exactly, it is not in the literature, which leads flexibility for the extensions to the cases with disturbances and uncertainties in addition to clock offsets.

This paper is organized as follows. We formally state the problem in Section II and introduce the notion of tight state estimation sets in Section III. In Section IV, we consider the case where there is no quantization and present conditions for stabilizability under clock offsets. The setup is extended to the case of quantized measurements in Section V, and stabilization under both clock offsets and quantization is considered in Section VI. Finally, we provide concluding remarks in Section VII. A subset of the results in Section IV appeared in the conference paper [33].

Notations: $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbf{0}$ (bold-faced zero) stands for the zero vector of appropriate dimension, and $\|\cdot\|$ represents the Euclidean norm.

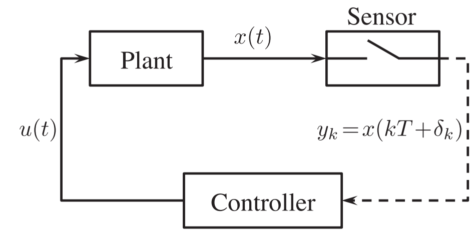

Fig. 1. Feedback system.

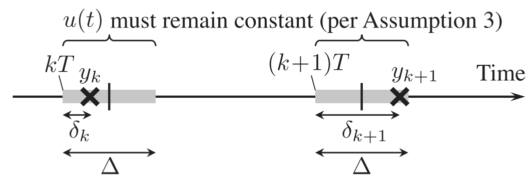

Fig. 2. Time chart.

II. Problem Formulation

Consider the feedback system depicted in Fig. 1. The plant to be controlled is the following continuous-time LTI process:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. The unknown initial state $x(0)$ may be anywhere in $\mathbb{R}^n$. We introduce the following assumption on the plant.

Assumption 1: The system (1) is unstable and controllable, and A is invertible.

It is common in the literature (e.g., [28], [29]) to consider unstable and controllable systems to make the stabilization problem nontrivial, as any component of the state that belongs to a stable invariant subspace of A will converge to zero, provided that the control signal converges to zero. We note that even if A is singular, that does not change the stabilizability of the plant for clock synchronization errors. Suppose that A has a zero eigenvalue and the corresponding state is decomposed. By letting the input to the state be zero during a sampling period, the state will be observed exactly. Once it is known, the controller can bring it to the origin because of controllability. Thus we assume that A is nonsingular.

The sensor attempts to sample the state periodically with period T, but does not necessarily achieve it due to clock errors. Let y_k, $k \in \mathbb{Z}_+$, be the kth observed state, and let $T > 0$ be the desired sampling period from the perspective of the controller clock. Fig. 2 illustrates a time chart of sampling: The actual time instants $kT + \delta_k$ at which the samples y_k are generated are represented by the crosses, whereas the gray boxes represent the uncertainty δ_k inherent to the timing information due to the offset between the clocks of the sensor and the controller. We consider the case in which the magnitude of the clock offset is bounded.

Assumption 2: For all time $k \in \mathbb{Z}_+$, δ_k is bounded as

$$0 \le \delta_k \le \Delta < T \tag{2}$$

with a constant Δ.

There are various algorithms for the clock offset estimation, and the resulting estimation errors have been analyzed; see, e.g., [34]. These results can be used to determine the offset bound Δ. The positiveness of the offset δ_k introduces no loss of generality, as it can always be guaranteed by an appropriate redefinition of the time scale. Under Assumption 2, the sequence y_k of observed states is produced by the following model:

$$y_k = x(kT + \delta_k), \quad \delta_k \in [0, \Delta] \tag{3}$$

where the actual sampling instant $kT + \delta_k$ is unknown because of the unknown jitter δ_k. We note that the controller cannot determine the precise value of δ_k based on the time instant at which the sample y_k arrives because of unknown communication delay.

Using the received observations $y_0, \ldots, y_k$, the controller needs to construct the input $u(t)$, $t \geq 0$. We consider the following assumption.

Assumption 3: The control input is constant during $[kT, kT + \Delta]$ for every k:

$$u(t) = u_k, \quad \forall t \in [kT, kT + \Delta] \quad \forall k \in \mathbb{Z}_+$$

and $u(t)$ does not depend on the future measurements $\{y_k : kT + \Delta \geq t\}$.

This assumption guarantees causality, but it also implies that during the intervals in which the sensor may measure the plant state, the controller holds the input. During the period $t \in (kT + \Delta, (k+1)T)$ for which the state has been measured, we allow the controller to take any value.

Our objective is to clarify how large Δ can be to ensure the stabilizability of the system.

Remark 1: Assumption 3 is required to prove the necessity results in the paper. We note that the sufficient conditions for stabilizability that we present in Sections IV and VI hold even if we limit the class of controllers to piecewise constant ones, for which Assumption 3 trivially holds.

III. Computation of Estimation Sets

In Sections III and IV, we consider a feedback system with no quantization and study the effect of clock mismatches between the sensor and the controller on the stabilizability of the system.

Here, we introduce the notion of a tight state estimation set, which will be instrumental to establish the necessity results in the paper. Let I_k^- denote the set of plant states $x(kT)$ that are compatible with all the measurements $y_0, y_1, \ldots, y_{k-1}$ taken *before* time kT. We shall call this the *estimation set of* $x(kT)$. Given I_k^-, we now study how to update the estimation set of $x(kT)$ using the new observation y_k. From (3), y_k is given by

$$y_k = \mathrm{e}^{A\delta_k}\left(x(kT) + \Psi(\delta_k)u_k\right) \tag{4}$$

where $\Psi(\delta_k) := \int_0^{\delta_k} \mathrm{e}^{-As} B \mathrm{d}s$. Thus, y_k depends on the unknown parameter $\delta_k \in [0, \Delta]$ and the true state $x(kT)$. Once y_k is known, $x(kT)$ must lie in the following set J_k:

$$J_k = \left\{\mathrm{e}^{-A\delta_k} y_k - \Psi(\delta_k)u_k : \delta_k \in [0, \Delta]\right\}. \tag{5}$$

Here, the second equality follows by Assumptions 1 and 3. With this set J_k, after receiving y_k, we obtain an updated estimation set I_k that now also considers the measurement y_k and can be defined by

$$I_k := I_k^- \cap J_k. \tag{6}$$

Since $I_0^- = \mathbb{R}^n$, (6) leads to $I_0 = J_0$. The estimation set I_{k+1}^- at the next time $x((k+1)T)$ is constructed from I_k using the model (1) and is given by

$$I_{k+1}^- = \left\{\mathrm{e}^{AT} x_k + \int_{kT}^{(k+1)T} \mathrm{e}^{A((k+1)T - \tau)} B u(\tau) \mathrm{d}\tau : x_k \in I_k\right\}. \tag{7}$$

The set I_k defined in (6) is a *tight* estimation set of $x(kT)$ in the following sense: For every $x_k \in I_k$, there exists a trajectory for the state $x(t)$, $t \in [0, kT + \Delta]$, that produces $y_0, y_1, \ldots, y_k$ with possible delays $\{\delta_i \in [0, \Delta]\}_{i=0}^k$ and satisfies $x(kT) = x_k$. We formally define tightness of an estimation set below.

Definition 1: An estimation set $I_k \subset \mathbb{R}^n$ of $x(kT)$, which is constructed from I_0^-, $u(t)$, $t \in [0, kT + \Delta]$, and $y_0, y_1, \ldots, y_k$, is said to be *tight* if for every $x_k \in I_k$, there exist $x_0 \in I_0^-$ and $\{\delta_i \in [0, \Delta]\}_{i=0}^k$, such that

$$\mathrm{e}^{AkT} x_0 + \int_0^{kT} \mathrm{e}^{A(kT - \tau)} B u(\tau) \mathrm{d}\tau = x_k \tag{8}$$

$$\mathrm{e}^{A(iT + \delta_i)} x_0 + \int_0^{iT + \delta_i} \mathrm{e}^{A(iT + \delta_i - \tau)} B u(\tau) \mathrm{d}\tau = y_i$$
$$\forall i \in \{0, 1, \ldots, k\}. \tag{9}$$

The following result is a direct consequence of the construction outlined above for estimation sets.

Proposition 1: The estimation set I_k defined in (6) is tight in the sense of Definition 1.

IV. Stabilizability With Infinite Bit-Rate

We start with the case where (1) is a process with a scalar-valued state and derive a tight bound on the maximum offset Δ for stabilizability. We then show that for a process with a vector-valued state case, clock offsets do not necessarily limit stabilizability.

We employ the following stabilizability definition.

Definition 2: The plant (1) is stabilizable if for any sequence of offsets $\{\delta_k\}_{k=0}^\infty$, $\delta_k \in [0, \Delta]$, and any initial state $x(0) \in I_0^-$, there exists a feedback controller such that the closed-loop state converges to the origin, i.e., $\lim_{t\to\infty} x(t) = \mathbf{0}$.

A. Process with a Scalar-Valued State

Consider the following controllable, unstable process where the state is scalar:

$$\dot{x}(t) = \lambda x(t) + b u(t). \tag{10}$$

From controllability in Assumption 1, we assume without loss of generality that $b = 1$ and, from instability, that $\lambda > 0$.

Theorem 1: Let Assumptions 1–3 hold. Then the plant (10) is stabilizable in the sense of Definition 2, if and only if the

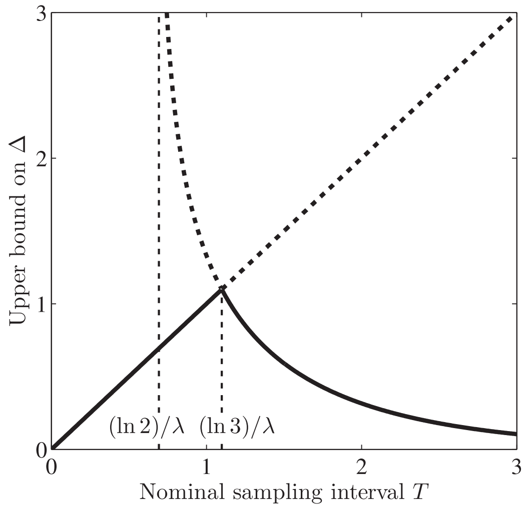

Fig. 3. Upper bound on Δ versus sampling period T ($\lambda = 1.0$): The solid line is the active bound $\bar{\Delta}$ and the vertical dotted lines indicate $T = (\ln 2)/\lambda$ (left) and $T = (\ln 3)/\lambda$ (right), respectively.

bound on the offset satisfies $0 \le \Delta < \bar{\Delta}$, where

$$\bar{\Delta} := \begin{cases} T & \text{if } 0 < T \le \frac{1}{\lambda}\ln 3 \\ T - \frac{1}{\lambda}\ln(\mathrm{e}^{\lambda T} - 2) & \text{if } T > \frac{1}{\lambda}\ln 3. \end{cases} \tag{11}$$

The bound $T - \{\ln(\mathrm{e}^{\lambda T} - 2)\}/\lambda$ in the lower branch of (11) is monotonically decreasing with respect to λ and T. This observation is consistent with the intuition that a larger growth rate $\mathrm{e}^{\lambda T}$ during one sampling period will result in a tighter requirement on the clock accuracy. The limitation $\bar{\Delta} = T$ for the case $0 < T < (\ln 3)/\lambda$ arises directly from Assumption 2. Fig. 3 shows the value of $\bar{\Delta}$ versus the sampling period T for the plant (10) with $\lambda = 1.0$.

Remark 2: In [30, Th. 14], a bound on clock offsets for stabilizability of a first-order system has also been given. Although the setup is slightly different from the present paper, it is worth noticing that both bounds decay exponentially with respect to the period T. In [30], the offsets are assumed to be static and the controller is taken as LTI. Thus, Theorem 1 provides the limitation compatible for a more general setup.

Our approach to prove Theorem 1 is based on the analysis of the estimation sets introduced in Section III. For the plant (10), by (5), the estimation set J_k computed from y_k can be written as

$$J_k = \left\{\mathrm{e}^{-\lambda\delta_k}\left(y_k + \lambda^{-1}u_k\right) - \lambda^{-1}u_k : \delta_k \in [0, \Delta]\right\}. \tag{12}$$

Thus, J_k becomes a bounded interval in $\mathbb{R}$. Therefore, with (6) and (7), I_k and I_{k+1}^-, $k \in \mathbb{Z}_+$, also result in bounded intervals in $\mathbb{R}$. Notice that we have assumed $I_0^- = \mathbb{R}$.

Let l_k be the length of the interval I_k, which is the updated estimation set of $x(kT)$ defined in (6). In the following, we study whether or not $l_k \to 0$ from which we will be able to conclude stabilizability. Suppose that I_{k-1} is obtained based on $y_0, y_1, \ldots, y_{k-1}$ at $t = (k-1)T + \Delta$. Then, one can compute I_k^- from (7). The length l_k is determined by I_k^- and J_k, thus, l_k depends on the true state $x_k \in I_k^-$, the jitter $\delta_k \in [0, \Delta]$, and the

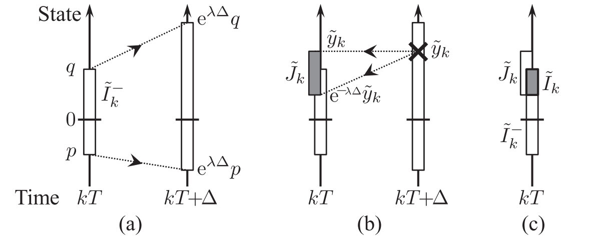

Fig. 4. Update procedure of the estimation set: (a) The initial estimation set $\tilde{I}_k^-$ expands during the sensing interval $[kT, kT + \Delta]$. (b) After receiving $\tilde{y}_k$, the controller computes the set $\tilde{J}_k$ of the candidates of the state at kT which may result in $\tilde{y}_k$ somewhere in $[kT, kT + \Delta]$. (c) The updated estimation set $\tilde{I}_k$ is obtained as the intersection of $\tilde{I}_k^-$ and $\tilde{J}_k$.

input $u(t)$, $t \in ((k-1)T + \Delta, kT)$. To prove that we are able to stabilize the closed loop for all possible x_k and δ_k, we have to consider the worst case trajectory. Hence, we need to determine under which conditions we have

$$\min_{u(t), t\in((k-1)T+\Delta, kT)} \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \not\to 0 \text{ as } k \to \infty$$

from which we would conclude that no controller could result in a decrease of the interval I_k to a single point and therefore that there are trajectories for which x_k does not converge. The following lemma is the key technical result needed to prove Theorem 1 as it gives the expansion rate of l_k.

Lemma 1: Given an estimation set I_{k-1} with width l_{k-1}, for every $k \ge 1$, it follows that

$$\min_{u(t), t\in((k-1)T+\Delta, kT)} \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k = \frac{(1 - \mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}}{2} l_{k-1} \tag{13}$$

where the minimum is achieved with any input that places the interval $\left\{x_k + \lambda^{-1}u_k : x_k \in I_k^-\right\}$ symmetrically about the origin.

Proof: For simplicity of notation, we first introduce biased sets as follows: Define $\tilde{y}_k := y_k + \lambda^{-1}u_k$ and let $\tilde{J}_k := \{x + \lambda^{-1}u_k : x \in J_k\}$. Then, from (12), this biased estimation set $\tilde{J}_k$ can be simply represented as

$$\tilde{J}_k = \begin{cases} \left[\mathrm{e}^{-\lambda\Delta}\tilde{y}_k,\ \tilde{y}_k\right] & \text{if } \tilde{y}_k > 0 \\ \{0\} & \text{if } \tilde{y}_k = 0 \\ \left[\tilde{y}_k,\ \mathrm{e}^{-\lambda\Delta}\tilde{y}_k\right] & \text{if } \tilde{y}_k < 0. \end{cases} \tag{14}$$

Similarly, we define the biased sets

$$\tilde{I}_k^- := \left\{x_k + \lambda^{-1}u_k : x_k \in I_k^-\right\}$$
$$\tilde{I}_k := \tilde{I}_k^- \cap \tilde{J}_k = \left\{x_k + \lambda^{-1}u_k : x_k \in I_k\right\}.$$

Note that the length l_k of I_k is the same as that of $\tilde{I}_k$ and the analysis in this proof holds for any bias $\lambda^{-1}u_k$. In Fig. 4, we illustrate the update procedure for the estimation set.

We now show that the input $u(t)$, $t \in ((k-1)T + \Delta, kT)$, achieving the minimum in (13) is the one that makes $\tilde{I}_k^-$ symmetric about the origin. In what follows, we evaluate the worst case length $\max_{x_k, \delta_k} l_k$ for a given $\tilde{I}_k^-$. We start by describing the range of y_k under the knowledge that $x_k \in I_k^-$ and $\delta_k \in [0, \Delta]$.

With (4), the set of all possible y_k is as follows:

$$y_k \in \left\{ \mathrm{e}^{\lambda\delta_k} x_k + \lambda^{-1}(\mathrm{e}^{\lambda\delta_k} - 1)u_k : x_k \in I_k^-,\ \delta_k \in [0, \Delta] \right\}$$

which results in

$$\tilde{y}_k \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0, \Delta],\ z \in \tilde{I}_k^- \right\}. \tag{15}$$

Thus, we can rewrite the maximization term in (13) as

$$\max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k = \max_{\tilde{y}_k \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0,\Delta], z \in \tilde{I}_k^- \right\}} l_k.$$

Next, we evaluate $\max_{\tilde{y}_k} l_k$ for each variation of $\tilde{I}_k^-$, and then compute its minimum over $u(t), t \in ((k-1)T + \Delta, kT)$. In the rest of the proof, we assume $\tilde{y}_k \neq 0$ since it follows from (14) that $\tilde{y}_k = 0$ does not maximize l_k. Denote the lower bound and the upper bound of $\tilde{I}_k^-$ by p and q, respectively: $\tilde{I}_k^- = [p, q]$. Suppose that

$$|p| \leq |q| \tag{16}$$

i.e., the midpoint of $\tilde{I}_k^-$ is nonnegative. For the case $|p| > |q|$, one can apply the same discussion below by flipping signs.

Consider the following two cases.

1) $p \leq 0$: In this case, it follows that $q \geq 0$ from (16) and thus, $\tilde{I}_k^-$ contains the origin. By (14), we have

$$\tilde{I}_k = \begin{cases} \left[\mathrm{e}^{-\lambda\Delta}\tilde{y}_k, \min(q, \tilde{y}_k)\right] & \text{if } \tilde{y}_k > 0 \\ \left[\max(p, \tilde{y}_k), \mathrm{e}^{-\lambda\Delta}\tilde{y}_k\right] & \text{if } \tilde{y}_k < 0. \end{cases}$$

Taking maximums in both cases $\tilde{y}_k > 0$ and $\tilde{y}_k < 0$, and using (16), one can see that the length l_k is maximum when $\tilde{y}_k = q$

$$\max_{\tilde{y}_k \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0,\Delta], z \in \tilde{I}_k^- \right\}} l_k = (1 - \mathrm{e}^{-\lambda\Delta})q.$$

The right-hand side takes its minimum under (16) and 1) for the case in which $q = -p = \mathrm{e}^{\lambda T} l_{k-1}/2$, and the minimum is

$$\min_{u(t), t \in ((k-1)T + \Delta, kT)} \ \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k = (1 - \mathrm{e}^{-\lambda\Delta}) \frac{\mathrm{e}^{\lambda T}}{2} l_{k-1}. \tag{17}$$

2) $p > 0$: In this case, noticing that all possible $\tilde{y}_k$ are positive from (15), we obtain

$$\tilde{I}_k = \left[\max(p, \mathrm{e}^{-\lambda\Delta}\tilde{y}_k), \min(q, \tilde{y}_k)\right].$$

Therefore

$$\max_{\tilde{y}_k \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0,\Delta], z \in \tilde{I}_k^- \right\}} l_k = \begin{cases} (1 - \mathrm{e}^{-\lambda\Delta})q & \text{if } \mathrm{e}^{-\lambda\Delta} q \geq p \\ q - p & \text{if } \mathrm{e}^{-\lambda\Delta} q < p. \end{cases} \tag{18}$$

From the condition 2) and (7), we have $q > q - p = \mathrm{e}^{\lambda T} l_{k-1}$. Thus, both branches in (18) are greater than the right-hand side of (17). Hence, $\tilde{I}_k^-$ satisfying 2) cannot minimize $\max_{\tilde{y}_k} l_k$, which concludes the proof of Lemma 1. ■

Proof of Theorem 1: (Sufficiency) If $\Delta < \bar{\Delta}$, then we have that $\gamma := (1 - \mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}/2 < 1$. Thus, from Lemma 1, for a given estimation set I_{k-1} of $x((k-1)T)$, there exists a control input resulting in $l_k < \gamma l_{k-1}$ for all δ_k and $x_{k-1} \in I_{k-1}$. Such control input can be constructed as follows: Divide the interval $((k-1)T + \Delta, kT)$ into two parts of equal length. In the first time period, the controller applies a constant input which makes I_k^- symmetric about $\lambda^{-1}u_k$. As we see in the proof of Lemma 1, this control input minimizes $\max_{x_k, \delta_k} l_k$, which is the length of the estimation set I_k in the worst case. In the second time period, the control input is simply set to 0. This control input satisfies Assumption 3 and simplifies the calculation of the estimation set J_k given by (5). Repeating this procedure for each sampling period, we can make the sequence l_k converge to 0, which implies the stability of the feedback system.

(Necessity) When $\gamma := (1 - \mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}/2 \geq 1$, the sequence l_k does not converge to zero, no matter how we choose $u(t)$. This means that there always exist a positive constant ϵ and a sequence of integers $k_1, k_2, \ldots$ such that $l_{k_i} \geq \epsilon, \forall i$. Since I_k is a tight estimation set, this means that there exists a possible trajectory of $x(t)$ such that $|x(t_{k_i})| \geq \epsilon/2, \forall i$ and therefore $x(t)$ does not converge to zero. ■

We now present a numerical example to illustrate the behavior of the system.

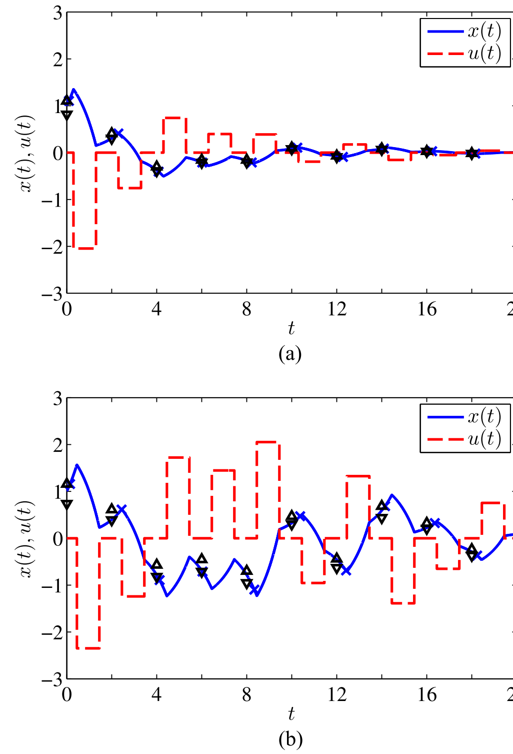

Fig. 5. Behavior of the system ($\lambda = 1.0$, $T = 2.0$, $\bar{\Delta} \approx 0.32$): The plant state $x(t)$ (solid), the input $u(t)$ (dashed), the observations y_k (cross), and the estimation sets I_k (△: upper bounds, ▽: lower bounds). (a) $\Delta = 0.30 < \bar{\Delta} \approx 0.32$. (b) $\Delta = 0.45 > \bar{\Delta} \approx 0.32$.

Example 1: Consider the system (10) with $\lambda = 1.0$ and the nominal sampling period $T = 2.0$. The initial state is taken to be $x(0) = 1$ and the δ_k are chosen uniformly randomly from $[0, \Delta]$. We employ the control law mentioned in the proof of Theorem 1, which minimizes the length of the worst case estimation set. We first consider the case of a small offset bound that satisfies the stability condition in Theorem 1: suppose $\Delta = 0.30$, which is less than $\bar{\Delta} \approx 0.32$. In Fig. 5(a), we illustrate the state $x(t)$ and the input $u(t)$ by the solid line and the dashed line, respectively. Moreover, the observations y_k are depicted by the cross marks and the upper and lower bounds of the estimation sets I_k are represented by $\triangle$ and $\triangledown$, respectively. One can observe that, as time progresses, the upper bounds and the lower bounds of I_k become closer, and the state successfully converges to 0. Next, we consider the case $\Delta = 0.45$, which violates the stability limit $\bar{\Delta} \approx 0.32$. Fig. 5(b) shows the behavior of the system in this case. We see that the state (solid line) oscillates and the estimation set I_k (triangle marks) does not shrink to a single point.

B. Process with a Vector-Valued State

In this section, we consider plants where A is not scalar.

Theorem 2: Let Assumptions 1–3 hold. If A has at least two distinct eigenvalues, then for any Δ satisfying (2), the plant (1) is stabilizable in the sense of Definition 2 and the state can be taken to the origin in a finite time.

Theorem 2 follows from the lemma below and the controllability of the system.

Lemma 2: If A has at least two distinct eigenvalues, then for any initial state and for any Δ satisfying (2), there exists a control input such that the state of (1) can be reconstructed precisely with a finite number of measurements.

Proof: See Section IV-C. ■

In view of Theorem 2, and unlike in the scalar case, there is no practical limitation on the clock offset if the plant dynamics has two distinct eigenvalues. Intuitively, we can explain this result as follows: At the time after receiving the measurement $y_k = x(kT + \delta_k)$, the controller has $n + k + 1$ unknown variables, i.e., $x(0)$ and $\delta_0, \ldots, \delta_k$. On the other hand, the measurements $y_0, y_1, \ldots, y_k$ provide $2(k + 1)$ equations that constrain these unknowns. If these equations were linear and independent, then one could determine the exact state in k steps as soon as $2(k + 1) \geq n + k + 1$. In reality they are not, but when A has two distinct eigenvalues, using an appropriate input signal $u(t)$ they provide enough "independence" to recover the initial state. Moreover, since the plant is controllable, once its state is precisely known, there exists an input that drives the state to the origin in a finite time interval.

We now show an example where I_1 becomes a single point. In this example, we can make the state converge to zero using only two observations y_0 and y_1.

Example 2: Consider the following system:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T = 1.5, \quad \Delta = 1.0.$$

Notice that Δ is greater than the bound (11) in the scalar case for each eigenvalue, 2 and 1, of A since $T - \{\ln(\mathrm{e}^{2T} - 2)\}/2 \approx 0.052$ and $T - \ln(\mathrm{e}^{T} - 2) \approx 0.59$. In Fig. 6, we plot the estimation sets I_0, I_1^-, and J_1 computed based on y_0 and y_1 in the state space $\mathbb{R}^2$. Since we assume $I_0^- = \mathbb{R}^2$, I_0 is equal to J_0. Note that the unknown states $x(0)$ and $x(T)$ are not used to compute the estimation sets. The figure shows that $I_1^- \cap J_1 = I_1$, which contains a single possible value for $x(T)$.

C. Proof of Lemma 2

As a preliminary of the proof, we first introduce a subsystem of (1) associated with the two distinct eigenvalues. If there

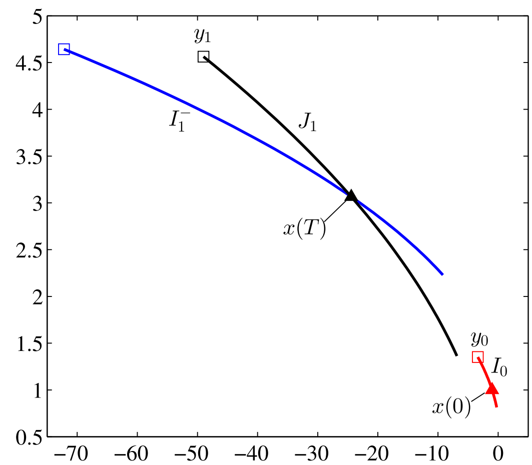

Fig. 6. Estimation sets I_0, I_1^-, and J_1 in the state space $\mathbb{R}^2$ (horizontal axis: first element of x, vertical axis: second element of x).

exist eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ of A such that $\lambda_1 \neq \lambda_2$, via an appropriate coordinate transformation, we have

$$V^{-1}\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 & \mathbf{0} \\ 0 & \lambda_2 & \mathbf{0} \\ * & * & * \end{bmatrix} V^{-1}x(t) + V^{-1}Bu(t)$$

for some nonsingular matrix $V \in \mathbb{R}^{n\times n}$; otherwise there exist $\lambda, \theta \in \mathbb{R}$ with $\theta \neq 0$ such that the complex conjugate pair $\lambda + i\theta, \lambda - i\theta$ are eigenvalues of A, and via an appropriate coordinate transformation, we have

$$V^{-1}\dot{x}(t) = \begin{bmatrix} \lambda & \theta & \mathbf{0} \\ -\theta & \lambda & \mathbf{0} \\ * & * & * \end{bmatrix} V^{-1}x(t) + V^{-1}Bu(t)$$

for some nonsingular matrix $V \in \mathbb{R}^{n\times n}$ [35, pp. 152–153]. Extracting the first two rows of the above system, we obtain the subsystem

$$\dot{\xi}(t) = \Lambda\xi(t) + \nu(t), \quad \zeta_k = \xi(kT + \delta_k) \tag{19}$$

where ξ and ν are the corresponding states and the inputs, respectively, and ζ_k is the output, i.e., the first two elements of $V^{-1}y_k$. For the case of two distinct real eigenvalues λ_1, λ_2, the matrix $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2)$; for the case of a complex conjugate pair of eigenvalues $\lambda + i\theta, \lambda - i\theta$, the matrix

$$\Lambda := \begin{bmatrix} \lambda & \theta \\ -\theta & \lambda \end{bmatrix}.$$

For the first step of the proof, we compute estimation sets for the state $\xi(kT)$ based on the measurements $y_0, \ldots, y_k$. Instead of the estimation sets of x represented by J_k, I_k, and I_{k+1}^-, which are defined in (5)–(7), we denote the corresponding estimation sets of ξ by $\hat{J}_k$, $\hat{I}_k$, and $\hat{I}_{k+1}^-$. In light of $I_0 = J_0$ and (7), the estimation set $\hat{I}_1^-$ of $\xi(T)$ is expressed as

$$\hat{I}_1^- = \left\{ \mathrm{e}^{-\Lambda\delta_0}\hat{a}_0 + \hat{b}_0 : \delta_0 \in [0, \Delta] \right\}.$$

Here, $\hat{a}_0$ and $\hat{b}_0$ are defined as follows: Let ν_k be the first two elements of $V^{-1}Bu_k$. Then, $\hat{a}_0$ and $\hat{b}_0$ are defined as

$$\hat{a}_0 := \mathrm{e}^{\Lambda T}(\zeta_0 + \Lambda^{-1}\nu_0)$$
$$\hat{b}_0 := -\mathrm{e}^{\Lambda T}\Lambda^{-1}\nu_0 + \int_0^T \mathrm{e}^{\Lambda(T-\tau)}\nu(\tau)\mathrm{d}\tau. \tag{20}$$

Note that Λ is nonsingular since A is invertible. Moreover, by (5), it follows that

$$\hat{J}_1 = \left\{\mathrm{e}^{-\Lambda\delta_1}\hat{a}_1 + \hat{b}_1 : \delta_1 \in [0, \Delta]\right\}$$

where

$$\hat{a}_1 := \zeta_1 + \Lambda^{-1}\nu_1, \quad \hat{b}_1 := -\Lambda^{-1}\nu_1. \tag{21}$$

1) Finiteness of $\hat{I}_1$: The first step of the proof is to show that there exists a control input $\nu(t), t \in [0, T+\Delta]$, that makes the updated estimation set $\hat{I}_1 = \hat{I}_1^- \cap \hat{J}_1$ contain at most a finite number of points. To do so, we need to consider the following problem: Let a, b, and c be nonzero vectors in $\mathbb{R}^2$; also fix $[T_1, T_2] \subset \mathbb{R}$ and $[T_3, T_4] \subset \mathbb{R}$. Define functions $f : [T_1, T_2] \to \mathbb{R}^2$ and $g : [T_3, T_4] \to \mathbb{R}^2$ as

$$f(t) := \mathrm{e}^{\Lambda t}a, \quad g(t) := \mathrm{e}^{\Lambda t}b + c \tag{22}$$

and find the intersection of $\{f(t) : t \in [T_1, T_2]\}$ and $\{g(t) : t \in [T_3, T_4]\}$. The following lemma states that this intersection can be a set of finite points if a, b, and c are chosen appropriately.

Lemma 3: If at least one of the vectors a, b, and c is not an eigenvector of Λ, then the intersection of the sets $\mathcal{F} := \{f(t) : t \in [T_1, T_2]\}$ and $\mathcal{G} := \{g(t) : t \in [T_3, T_4]\}$ contains, at most, a finite number of points.

Proof: See Section IV-D. ∎

Consider the statement of Lemma 3 with $a = \hat{a}_0, b = \hat{a}_1$, and $c = \hat{b}_0 - \hat{b}_1$. Then, $\hat{I}_1$ corresponds to the intersection of $\mathcal{F}$ and $\mathcal{G}$. Note that $\hat{a}_0$ and $\hat{a}_1$ can be assumed to be nonzero since if either $\hat{a}_0 = \mathbf{0}$ or $\hat{a}_1 = \mathbf{0}$, then $\hat{I}_1^-$ or $\hat{J}_1$ becomes a single point, and the result is trivially true.

In what follows, we show that there exists an input such that $\hat{b}_0 - \hat{b}_1$ is not equal to $\mathbf{0}$ nor is an eigenvector of Λ. From Assumption 3, $\nu(t)$ takes a constant value ν_0 for $t \in [0, \Delta]$, thus, it follows from (20) and (21) that

$$\hat{b}_0 - \hat{b}_1 = -\Lambda^{-1}\mathrm{e}^{\Lambda(T-\Delta)}\nu_0 + \int_\Delta^T \mathrm{e}^{\Lambda(T-\tau)}\nu(\tau)\mathrm{d}\tau + \Lambda^{-1}\nu_1.$$

Note that the first and the third terms in the right-hand side are known to the controller. Furthermore, the integral term is the zero-state-response of (19). Because of the controllability of the original system (1), $x(kT)$ or $\xi(kT)$ can be set arbitrarily, and hence $\hat{b}_0 - \hat{b}_1$ can be made an arbitrary vector by selecting an appropriate input $\nu(t), t \in (\Delta, T)$. Thus, with an input $\nu(t)$ such that $\hat{b}_0 - \hat{b}_1$ is not $\mathbf{0}$ nor an eigenvector of Λ, it follows from Lemma 3 that $\hat{I}_1$ becomes a set of finite points.

2) Stabilizing the System with the Finite Set $\hat{I}_1$: The second step of the proof consists of providing a procedure to determine the value of the state once $\hat{I}_1$ contains only a finite number of points and bring it to the origin. Consider the subsystem (19) of (1). From the first step, we can pick the control signal so that $\hat{I}_1$ is a set with a finite number of points. For each element of $\hat{I}_1$, we can determine corresponding δ_1 and $x(2T)$ as follows. Pick a point $\xi_1 \in \hat{I}_1$. From (19), we have that

$$\zeta_1 = \mathrm{e}^{\Lambda\delta_1}\xi_1 + \int_0^{\delta_1}\mathrm{e}^{\Lambda(\delta_1-\tau)}\nu(\tau+T)\mathrm{d}\tau$$
$$= \mathrm{e}^{\Lambda\delta_1}\left(\xi_1 + \Lambda^{-1}\nu_1\right) - \Lambda^{-1}\nu_1.$$

Thus, if $\xi_1 \neq -\Lambda^{-1}\nu_1$, δ_1 is uniquely determined from variables known to the controller. When $\xi_1 = -\Lambda^{-1}\nu_1$, we select the input so that $\xi_2 := \mathrm{e}^{\Lambda T}\xi_1 + \int_0^T \mathrm{e}^{\Lambda(T-\tau)}\nu(\tau+T)\mathrm{d}\tau$ satisfies $\xi_2 \neq -\Lambda^{-1}\nu_2$, which is always possible in view of controllability. In this case, the set $\hat{I}_2$ would continue to be finite, and we would be able to determine the corresponding δ_2 using a similar procedure. For simplicity, in the remainder of the proof we shall assume that we have determined δ_1, but the reasoning would be similar if we had δ_2 instead. For the value of δ_1 so obtained, we can compute $x(T)$ by solving

$$y_1 = \mathrm{e}^{AT}x(T) + \int_0^{\delta_1}\mathrm{e}^{A(\delta_1-\tau)}Bu(\tau+T)\mathrm{d}\tau$$

and can compute $x(2T)$ using the variation of constants formula. From controllability, the controller can bring this point $x(2T)$ to the origin by an appropriate open-loop input $u(t)$ for $t \in (2T+\Delta, 3T)$. The next output y_3 takes a nonzero value only if the selected point ξ_1 that we used to estimate $x(2T)$ does not correspond to the true state trajectory. Hence, we reduce the number of elements of $\hat{I}_k$ at least by one element in one sampling interval (or two periods, in case we had to consider δ_2). Repeating this procedure, we eventually obtain $y_k = 0$, which means that the origin has been reached.

D. Proof of Lemma 3

The proof of Lemma 3 is based on the following corollary of the mean-value theorem.

Corollary 1: Consider two continuous functions $\tau_1, \tau_2 : \mathcal{S} \to \mathbb{R}$ defined on a finite interval $\mathcal{S} \subset \mathbb{R}$. Suppose there are two distinct points $s_1, s_2 \in \mathcal{S}$ such that

$$\tau_1(s_1) - \tau_2(s_1) = \tau_1(s_2) - \tau_2(s_2)$$

and τ_1, τ_2 are differentiable in (s_1, s_2). Then there exists a point $s^* \in (s_1, s_2)$ such that the derivatives

$$\tau_1'(s^*) = \tau_2'(s^*).$$

Proof: Define the function $\Delta_\tau : \mathcal{S} \to \mathbb{R}$ by $\Delta_\tau(s) := \tau_1(s) - \tau_2(s)$. Then $\Delta_\tau(s_1) = \Delta_\tau(s_2)$. Following the mean-value theorem, there exists $s^* \in (s_1, s_2)$ such that

$$\tau_1'(s^*) - \tau_2'(s^*) = \Delta_\tau'(s^*) = \frac{\Delta_\tau(s_2) - \Delta_\tau(s_1)}{s_2 - s_1} = 0.$$

∎

Define the set

$$\mathcal{U} := \{(t, s) \in [T_1, T_2] \times [T_3, T_4] : f(t) = g(s)\}. \tag{23}$$

Then $\mathcal{F} \cap \mathcal{G}$ is infinite only if $\mathcal{U}$ is infinite. In the following, we assume that $\mathcal{U}$ is infinite and prove the claim of Lemma 3 by contradiction.

1) Case of Two Distinct Real Eigenvalues: From invertibility in Assumption 1, we assume without loss of generality that λ_1 and λ_2 are both nonzero. Denote

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

with scalars $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. From

$$\mathrm{e}^{\Lambda t} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} \end{bmatrix}$$

it follows that

$$f(t) = \begin{bmatrix} a_1 e^{\lambda_1 t} \\ a_2 e^{\lambda_2 t} \end{bmatrix}, \quad g(t) = \begin{bmatrix} b_1 e^{\lambda_1 t} + c_1 \\ b_2 e^{\lambda_2 t} + c_2 \end{bmatrix}.$$

For the set $\mathcal{U}$ defined by (23), $(t, s) \in \mathcal{U}$ if and only if

$$a_1 \mathrm{e}^{\lambda_1 t} = b_1 \mathrm{e}^{\lambda_1 s} + c_1, \quad a_2 \mathrm{e}^{\lambda_2 t} = b_2 \mathrm{e}^{\lambda_2 s} + c_2. \tag{24}$$

In the following, we transform (24) into two formulas for t in terms of s, and show that there are only a finite number of points where their values are equal.

First, we show that the scalars a_1, a_2, b_1, and b_2 are all nonzero. Suppose there exists an $i \in \{1, 2\}$ such that $a_i = 0$.

1) If $b_i = 0$, then $c_i \neq 0$ (otherwise the vectors a, b, and c are all eigenvectors of Λ). Thus, (24) implies that $\mathcal{U}$ is empty, which contradicts the assumption that it is infinite.
2) If $b_i \neq 0$, then $(t, s) \in \mathcal{U}$ only if $b_i c_i < 0$ and
$$s = \frac{1}{\lambda_i} \ln\left(-\frac{c_i}{b_i}\right).$$
Thus $\mathcal{U}$ is a singleton, which contradicts the assumption that it is infinite.

Hence a_1 and a_2 are both nonzero. The scalars b_1 and b_2 are both nonzero for similar reasons.

Second, we discard some subsets of $[T_3, T_4]$ where (24) cannot hold. Let $\bar{f}_0 := \min\{\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t} : t \in [T_1, T_2]\} > 0$. Define the functions $\bar{g}_1, \bar{g}_2 : [T_3, T_4] \to \mathbb{R}$ as

$$\bar{g}_i(s) := \frac{b_i \mathrm{e}^{\lambda_i s} + c_i}{a_i}, \quad i = 1, 2$$

and the sets $\mathcal{S}_1, \mathcal{S}_2 \subset [T_3, T_4]$ as

$$\mathcal{S}_i := \{s \in [T_3, T_4] : \bar{g}_i(s) \geq \bar{f}_0\}, \quad i = 1, 2.$$

As $\bar{g}_1$ and $\bar{g}_2$ are both monotonic functions, the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are both finite intervals on $\mathbb{R}$. Hence their intersection

$$\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2 \subset [T_3, T_4]$$

is also a finite interval on $\mathbb{R}$. Following (24), $(t, s) \in \mathcal{U}$ only if $s \in \mathcal{S}$, since otherwise there exits an $i \in \{1, 2\}$ such that

$$\frac{b_i \mathrm{e}^{\lambda_i s} + c_i}{a_i} = \bar{g}_i(s) < \bar{f}_0 \leq e^{\lambda_i t} \quad \forall t \in [T_1, T_2].$$

Finally, define the functions $\tau_1, \tau_2 : \mathcal{S} \to \mathbb{R}$ by

$$\tau_i(s) := \frac{1}{\lambda_i} \ln\left(\frac{b_i \mathrm{e}^{\lambda_i s} + c_i}{a_i}\right), \quad i = 1, 2.$$

Note that for both $i \in \{1, 2\}$

$$\frac{b_i \mathrm{e}^{\lambda_i s} + c_i}{a_i} = \bar{g}_i(s) \geq \bar{f}_0 > 0 \qquad \forall s \in \mathcal{S}.$$

Hence τ_1 and τ_2 are both well defined and continuous on $\mathcal{S}$. Also, they are both differentiable on the interior of $\mathcal{S}$, and their derivatives are given by

$$\tau_i'(s) = \frac{b_i \mathrm{e}^{\lambda_i s}}{b_i \mathrm{e}^{\lambda_i s} + c_i}, \quad i = 1, 2.$$

Following (24), $(t, s) \in \mathcal{U}$ only if $\tau_1(s) = \tau_2(s)$. Suppose there are two distinct points $s_1, s_2 \in \mathcal{S}$ such that $\tau_1(s_1) = \tau_2(s_1)$ and $\tau_1(s_2) = \tau_2(s_2)$. By Corollary 1, there exists a point $s^* \in (s_1, s_2)$ such that $\tau_1'(s^*) = \tau_2'(s^*)$, that is,

$$b_1 c_2 \mathrm{e}^{\lambda_1 s^*} = b_2 c_1 \mathrm{e}^{\lambda_2 s^*}.$$

As b_1 and b_2 are both nonzero, and c_1 and c_2 are not both zero, the previous equality holds if only if $b_1 b_2 c_1 c_2 > 0$, and

$$s^* = \frac{1}{\lambda_1 - \lambda_2} \ln\left(\frac{b_2 c_1}{b_1 c_2}\right). \tag{25}$$

We have thus shown that given any two points s_1 and s_2 in the set $\mathcal{V} := \{s \in \mathcal{S} : \tau_1(s) = \tau_2(s)\}$ the point s^* defined by (25) must be between those two points. This automatically excludes the possibility of $\mathcal{V}$ having three or more points, by a contradiction argument. Consequently, there are at most two points in $\mathcal{U}$, which contradicts the assumption that it is infinite. Therefore, $\mathcal{F} \cap \mathcal{G}$ is finite, that is, the claim of Lemma 3 holds for the case of two distinct real eigenvalues.

2) Case of a Complex Conjugate Pair of Eigenvalues: Denote

$$a = \begin{bmatrix} \hat{a} \cos \alpha \\ \hat{a} \sin \alpha \end{bmatrix}, \quad b = \begin{bmatrix} \hat{b} \cos \beta \\ \hat{b} \sin \beta \end{bmatrix}, \quad c = \begin{bmatrix} \hat{c} \cos \gamma \\ \hat{c} \sin \gamma \end{bmatrix}$$

with scalars $\hat{a}, \hat{b}, \hat{c} > 0$ and $\alpha, \beta, \gamma \in [0, 2\pi)$. From

$$\mathrm{e}^{\Lambda t} = \begin{bmatrix} \mathrm{e}^{\lambda t} \cos(\theta t) & \mathrm{e}^{\lambda t} \sin(\theta t) \\ -\mathrm{e}^{\lambda t} \sin(\theta t) & \mathrm{e}^{\lambda t} \cos(\theta t) \end{bmatrix}.$$

it follows that

$$f(t) = \begin{bmatrix} \hat{a} \mathrm{e}^{\lambda t} \cos(\alpha - \theta t) \\ \hat{a} \mathrm{e}^{\lambda t} \sin(\alpha - \theta t) \end{bmatrix}$$

$$g(t) = \begin{bmatrix} \hat{b} \mathrm{e}^{\lambda t} \cos(\beta - \theta t) + \hat{c} \cos \gamma \\ \hat{b} \mathrm{e}^{\lambda t} \sin(\beta - \theta t) + \hat{c} \sin \gamma \end{bmatrix}.$$

For brevity, denote

$$\begin{aligned} r_1(s) &:= \hat{b} \mathrm{e}^{\lambda s} \sin(\beta - \theta s) + \hat{c} \sin \gamma \\ r_2(s) &:= \hat{b} \mathrm{e}^{\lambda s} \cos(\beta - \theta s) + \hat{c} \cos \gamma. \end{aligned} \tag{26}$$

For the set $\mathcal{U}$ defined by (23), $(t, s) \in \mathcal{U}$ if and only if

$$\begin{aligned} \hat{a}^2 \mathrm{e}^{2\lambda t} &= r_1(s)^2 + r_2(s)^2 \\ \tan(\alpha - \theta t) &= r_1(s) / r_2(s). \end{aligned} \tag{27}$$

Consider the special case that $\lambda = 0$ (namely, a conjugate pair of purely imaginary eigenvalues). Then the first equation in (27) becomes

$$\hat{a}^2 = \hat{b}^2 + 2\hat{b}\hat{c}\cos(\beta - \gamma - \theta s) + \hat{c}^2.$$

As $\hat{a}, \hat{b}, \hat{c} > 0$, the previous equality may hold for only a finite number of points on the finite interval $[T_3, T_4]$, which contradicts the assumption that $\mathcal{U}$ is infinite.

In the following, we transform (27) into two formulas for t in terms of s (under the assumption that $\lambda \neq 0$), and show that there is only a finite number of points where their values are equal.

First, based on the periodicity of the trigonometric functions in (27), we divide the domains $[T_1, T_2]$ and $[T_3, T_4]$ into finitely many intervals. Define the sequences of finite intervals $(\mathcal{T}_l)_{l\in\mathbb{Z}}$ and $(\bar{\mathcal{S}}_l)_{l\in\mathbb{Z}}$ as

$$\mathcal{T}_l := \left[\frac{\alpha + l\pi}{\theta} - \frac{\pi}{2|\theta|}, \frac{\alpha + l\pi}{\theta} + \frac{\pi}{2|\theta|}\right)$$
$$\bar{\mathcal{S}}_l := \left(\frac{\beta + l\pi}{\theta}, \frac{\beta + l\pi}{\theta} + \frac{\pi}{|\theta|}\right).$$

Then

$$\bigcup_{l\in\mathbb{Z}} \mathcal{T}_l = \mathbb{R}, \quad \bigcup_{l\in\mathbb{Z}} \bar{\mathcal{S}}_l = \mathbb{R}\backslash\{s \in \mathbb{R} : \sin(\beta - \theta s) = 0\}.$$

As $[T_1, T_2]$ and $[T_3, T_4]$ are both finite, there are only a finite number of integers l_1 and l_2 such that $[T_1, T_2] \cap \mathcal{T}_{l_1} \neq \emptyset$ and $[T_3, T_4] \cap \bar{\mathcal{S}}_{l_2} \neq \emptyset$, respectively. Moreover, there are also only a finite number of points in $\{s \in [T_3, T_4] : \sin(\beta - \theta s) = 0\}$.

Second, we further divide the intervals in $(\bar{\mathcal{S}}_l)_{l\in\mathbb{Z}}$ by discarding the points where r_2 defined in (26) vanishes (to ensure the continuity of the function τ_2 below). As r_2 is continuous and monotonic on each $\bar{\mathcal{S}}_l$, there is at most one point $s_l' \in \bar{\mathcal{S}}_l$ such that $r_2(s_l') = 0$. If such a point exists then denote

$$\mathcal{S}_{2l} := \left(\frac{\beta + l\pi}{\theta}, s_l'\right), \quad \mathcal{S}_{2l+1} := \left(s_l', \frac{\beta + l\pi}{\theta} + \frac{\pi}{|\theta|}\right)$$

otherwise let

$$\mathcal{S}_{2l} := \mathcal{S}_{2l+1} := \bar{\mathcal{S}}_l.$$

Then $(\mathcal{S}_l)_{l\in\mathbb{Z}}$ is also a sequence of finite intervals. Again, as $[T_3, T_4]$ is finite, there are only a finite number of integers l such that $[T_3, T_4] \cap \mathcal{S}_l \neq \emptyset$. Therefore, if for each pair of integers l and m, there are only a finite number of points $(t, s) \in \mathcal{T}_l \times \mathcal{S}_m$ such that (27) holds, then $\mathcal{U}$ is also finite.

Finally, fix an arbitrary pair of integers l and m, and define the functions $\tau_1, \tau_2 : \mathcal{S}_m \to \mathbb{R}$ by

$$\tau_1(s) := \frac{1}{2\lambda}\ln\left(\frac{r_1^2(s) + r_2^2(s)}{\hat{a}^2}\right)$$
$$\tau_2(s) := \frac{1}{\theta}\left(\alpha + l\pi - \arctan\left(\frac{r_1(s)}{r_2(s)}\right)\right).$$

As r_1 and r_2 are both differentiable and r_2 is nonzero on $\mathcal{S}_m$, the functions τ_1 and τ_2 are both well defined and differentiable on $\mathcal{S}_m$. Also, as the image of the arctangent function on $\mathbb{R}$ is a subset of the interval $(-\pi/2, \pi/2)$, the image of τ_2 is a subset of $\mathcal{T}_l$. The derivatives of τ_1 and τ_2 are given by

$$\tau_1'(s) = \frac{\hat{b}^2 e^{2\lambda s} + \hat{b}\hat{c}e^{\lambda s}\cos(\phi(s)) + \hat{b}\hat{c}e^{\lambda s}(\theta/\lambda)\sin(\phi(s))}{r_1^2(s) + r_2^2(s)}$$
$$\tau_2'(s) = \frac{\hat{b}^2 e^{2\lambda s} + \hat{b}\hat{c}e^{\lambda s}\cos(\phi(s)) - \hat{b}\hat{c}e^{\lambda s}(\lambda/\theta)\sin(\phi(s))}{r_1^2(s) + r_2^2(s)}$$

where

$$\phi(s) := \beta - \gamma - \theta s.$$

For all $(t, s) \in \mathcal{T}_l \times \mathcal{S}_m$, (27) holds only if $\tau_1(s) = \tau_2(s)$. Suppose there are two distinct points $s_1, s_2 \in \mathcal{S}_m$ such that $\tau_1(s_1) = \tau_2(s_1)$ and $\tau_1(s_2) = \tau_2(s_2)$. By Corollary 1, there exists a point $s^* \in (s_1, s_2)$ such that $\tau_1'(s^*) = \tau_2'(s^*)$, that is,

$$\sin(\phi(s^*)) = \sin(\beta - \gamma - \theta s^*) = 0. \tag{28}$$

We have established that given any s_1, s_2 in the set $\mathcal{V}_m := \{s \in \mathcal{S}_m : \tau_1(s) = \tau_2(s)\}$ the point s^* satisfying (28) must be between s_1 and s_2. However, (28) holds for at most one point on $\mathcal{S}_m$. Hence, there are at most two distinct points in $\mathcal{V}_m$. Consequently, there are at most two points in $T_l \times S_m$ such that (27) holds, which contradicts the assumption that $\mathcal{U}$ is infinite (as explained in the second step above). Therefore, $\mathcal{F} \cap \mathcal{G}$ is finite, that is, the claim of Lemma 3 holds for the case of a complex conjugate pair of eigenvalues. This completes the proof of Lemma 3.

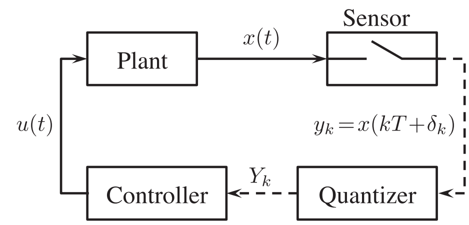

Fig. 7. Feedback system with quantizer.

V. Preliminaries for the Case with Quantization

So far, we have assumed that real-valued vectors y_k can be transmitted from the sensor to the controller. However, signals in networked control systems are often quantized to discrete values because of the sensor's limited capabilities and/or finite capacity in the communication channels. In the rest of the paper, we extend the setup in Section II to deal with quantization.

Instead of the original feedback system in Fig. 1, consider the system depicted in Fig. 7. The plant and controller are the same as those in Section II, but the observed state $y_k \in \mathbb{R}^n$ is quantized before being sent to the controller. We shall consider static quantization laws that partition the state-space $\mathbb{R}^n$ into a discrete family of sets and, at the kth sampling time transmit a symbol $r(y_k)$ that represents the set to which y_k belongs. For simplicity of notation, we denote by Y_k both the symbol and the set corresponding to the measurement y_k. In the next section, we present results for logarithmic and uniform quantization sets.

In the quantization case, we impose the following assumption instead of Assumption 3.

Assumption 4: The control input is zero during $[kT, kT + \Delta]$ for every k, i.e.,

$$u_k = \mathbf{0} \quad \forall k \in \mathbb{Z}_+$$

and $u(t)$ does not depend on the future measurements $\{Y_k : kT + \Delta \geq t\}$.

Remark 3: Assumption 4 simplifies computing the effect of the input u_k during the sensing period $[kT, kT + \Delta]$, while the exact value of y_k may not yet be known to the controller. This assumption can be lifted if the quantizer knows the control law.

We now redefine the estimation set J_k in (5) to fit the quantization setup:

$$J_k = \left\{ \mathrm{e}^{-A\delta_k} \hat{y}_k : \delta_k \in [0, \Delta],\ \hat{y}_k \in Y_k \right\}. \tag{29}$$

The estimation sets I_k and I_{k+1}^- are defined as in (6) and (7), respectively, but with J_k defined above instead of (5). Note that the estimation set I_k is also tight in the following sense, which is a natural extension of Definition 1.

Definition 3: An estimation set $I_k \subset \mathbb{R}^n$ of $x(kT)$, which is constructed from $I_0^-, u(t), t \in [0, kT + \Delta]$, and $Y_0, Y_1, \ldots, Y_k$, is said to be *tight* if for every $x_k \in I_k$, there exist $x_0 \in I_0^-$, $\{\delta_i \in [0, \Delta]\}_{i=0}^k$, and $\{\hat{y}_i \in Y_i\}_{i=0}^k$, such that

$$\begin{aligned} &\mathrm{e}^{AkT} x_0 + \int_0^{kT} \mathrm{e}^{A(kT-\tau)} Bu(\tau)\mathrm{d}\tau = x_k \\ &\mathrm{e}^{A(iT+\delta_i)} x_0 + \int_0^{iT+\delta_i} \mathrm{e}^{A(iT+\delta_i-\tau)} Bu(\tau)\mathrm{d}\tau = \hat{y}_i \\ &\forall i \in \{0, 1, \ldots, k\}. \end{aligned}$$

VI. Stabilizability Under Clock Offsets and Quantization

A. Process with a Scalar-Valued State: Logarithmic Quantizer

Consider the plant in (10). In this section, we employ the logarithmic quantizer proposed in [23]: Given $\alpha_0 > 0$ and $\rho \in (0, 1)$, the quantizer r is given by $Y_k = r(y_k), \forall k \in \mathbb{Z}_+$, with

$$r(y) = \begin{cases} \left(\rho^{i+1}\alpha_0, \rho^i \alpha_0\right] & \text{if } y > 0 \\ \{0\} & \text{if } y = 0 \\ \left[-\rho^i\alpha_0, -\rho^{i+1}\alpha_0\right) & \text{if } y < 0 \end{cases} \tag{30}$$

where i is the integer satisfying $\rho^{i+1}\alpha_0 < |y| \leq \rho^i \alpha_0$. Here, the ratio ρ expresses the coarseness of the logarithmic quantizer; the quantization levels become fine as $\rho \to 1$ and become coarse as $\rho \to 0$.

For the system given in Fig. 7 with this logarithmic quantizer, the following theorem gives a necessary condition and a sufficient condition for stability, expressed in terms of the parameter ρ that defines the coarseness of the quantizer.

Theorem 3: Let Assumptions 1, 2, and 4 hold. Consider the feedback system in Fig. 7 with the plant (10) and the logarithmic quantizer (30). If the system is stabilizable in the sense of Definition 2, then

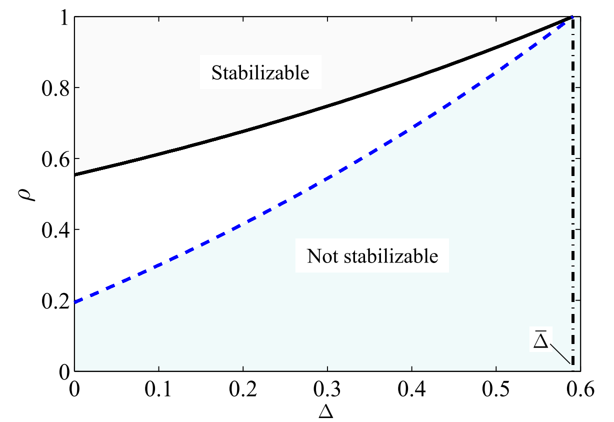

Fig. 8. Bounds on coarseness ρ of quantization for stabilizability versus the maximum clock offsets Δ: The necessary condition in (31) (dashed), the sufficient condition in (32) (solid), and the upper bound on the clock offset for stabilizability in (11) (dash–dot).

$$\rho \in \begin{cases} (0, 1) & \text{if } \mathrm{e}^{\lambda T} \leq 2 \\ \left(\max\left\{0, \mathrm{e}^{\lambda\Delta} - \frac{1}{\mathrm{e}^{\lambda T}/2 - 1}\right\}, 1\right) & \text{if } \mathrm{e}^{\lambda T} > 2. \end{cases} \tag{31}$$

On the other hand, the feedback system is stabilizable if

$$\rho \in \left(\max\left\{0, \mathrm{e}^{\lambda\Delta}\left(1 - \frac{1}{\mathrm{e}^{\lambda T}/2}\right)\right\}, 1\right). \tag{32}$$

We illustrate the bounds on the coarseness ρ given in (31) and (32) through an example.

Example 3: Consider the plant (10) with $\lambda = 1.0$ and fix the nominal sampling interval as $T = 1.5$. In Fig. 8, we plot the necessary lower bound on ρ for stabilizability given by (31) and the sufficient bound given by (32). The dash-dot line represents the maximum clock offset tolerable for stability $\bar{\Delta} \approx 0.59$ without quantization given by Theorem 1. We see that, as the clock offset Δ approaches $\bar{\Delta}$, the ratio ρ of the endpoints of a quantization cell goes to 1 and hence, very fine quantization is needed for stabilizability.

The approach used to prove Theorem 3 also relies on determining the length l_k of the estimation set I_k and construct upper and lower worst-case bounds for this length.

Lemma 4: Consider the feedback system in Fig. 7 with the plant (10) and a logarithmic quantizer (30). Given an estimation set I_{k-1} of width l_{k-1}, $k \geq 1$, the worst-case width l_k of I_k satisfies the following upper and lower bounds:

$$\begin{aligned} \frac{1 - \rho\mathrm{e}^{-\lambda\Delta}}{1 + (1-\rho)\mathrm{e}^{-\lambda\Delta}} \frac{\mathrm{e}^{\lambda T}}{2} l_{k-1} &\leq \min_{\substack{u(t),\\ t\in((k-1)T+\Delta, kT)}} \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \\ &\leq \left(1 - \rho\mathrm{e}^{-\lambda\Delta}\right) \frac{\mathrm{e}^{\lambda T}}{2} l_{k-1} \end{aligned} \tag{33}$$

where the second inequality holds when the control input places I_k^- symmetrically about the origin.

Proof: Let p and q denote the lower bound and the upper bound for I_k^-, respectively, i.e., $p := \inf_{\xi \in I_k^-} \xi$ and $q :=$

$\sup_{\xi\in I_k^-}\xi$. Assume that

$$|q| \geq |p| \tag{34}$$

that is, the center of I_k^- is nonnegative. If this is not the case, the analysis below can be adapted by flipping signs and replacing q with p.

As we discussed in the proof of Lemma 1, the min–max of l_k in (33) can be computed by finding the largest l_k, as we ranging over all possible quantized outputs Y_k. To obtain an explicit formula of $\max_{Y_k} l_k$, we consider the following two cases.

1) $p \leq 0 < q$: From (4), the set of possible Y_k compatible with $x_k \in I_k^-$ and $\delta_k \in [0,\Delta]$ is given by $Y_k \in \mathcal{Y}^- \cup \{0\} \cup \mathcal{Y}^+$, where $\mathcal{Y}^-$ and $\mathcal{Y}^+$ are defined as

$$\mathcal{Y}^- := \left\{ \left[-\rho^i\alpha_0, -\rho^{i+1}\alpha_0\right) : i \geq \left\lfloor \frac{\ln \frac{e^{\lambda\Delta}|p|}{\alpha_0}}{\ln\rho} \right\rfloor, i\in\mathbb{Z} \right\}$$

$$\mathcal{Y}^+ := \left\{ \left(\rho^{i+1}\alpha_0, \rho^i\alpha_0\right] : i \geq \left\lfloor \frac{\ln \frac{e^{\lambda\Delta}q}{\alpha_0}}{\ln\rho} \right\rfloor, i\in\mathbb{Z} \right\}.$$

We note that the interval $(\rho^{i+1}\alpha_0, \rho^i\alpha_0]$, $i = \lfloor \ln\frac{y}{\alpha_0}/\ln\rho\rfloor$, is the quantization cell that contains $y > 0$, i.e., $\rho^{i+1}\alpha_0 < y \leq \rho^i\alpha_0$.

In what follows, we evaluate $\max_{Y_k} l_k$ on each subset $\{0\}$, $\mathcal{Y}^+$, and $\mathcal{Y}^-$ to describe $\max_{Y_k\in\mathcal{Y}^-\cup\{0\}\cup\mathcal{Y}^+} l_k$. If $Y_k = \{0\}$, we have $I_k = \{0\}$ and hence $l_k = 0$. Next, suppose that $Y_k \in \mathcal{Y}^+$. Then, from (29), the estimation set J_k is given by

$$J_k = \left(e^{-\lambda\Delta}\rho^{i+1}\alpha_0, \rho^i\alpha_0\right], \quad i \geq \left\lfloor \frac{\ln \frac{e^{\lambda\Delta}q}{\alpha_0}}{\ln\rho} \right\rfloor, \quad i\in\mathbb{Z}$$

where, i represents the index of a quantization cell. Taking the intersection of J_k and I_k^-, which is bounded by q from above, we have

$$l_k = \begin{cases} q - e^{-\lambda\Delta}\rho^{i+1}\alpha_0 & \text{if } \rho^i\alpha_0 \geq q \\ \left(1 - e^{-\lambda\Delta}\rho\right)\rho^i\alpha_0 & \text{if } \rho^i\alpha_0 < q. \end{cases}$$

Let us consider the maximum of l_k over i. If $\rho^i\alpha_0 \geq q$, i.e., $i \leq \ln\frac{q}{\alpha_0}/\ln\rho$, then l_k is increasing with i, and hence the maximum in this case occurs when $i = \lfloor \ln\frac{q}{\alpha_0}/\ln\rho\rfloor$. If $\rho^i\alpha_0 < q$, or $i > \ln\frac{q}{\alpha_0}/\ln\rho$, since l_k is decreasing for this case, the maximum occurs when $i = \lfloor \ln\frac{q}{\alpha_0}/\ln\rho\rfloor + 1$. Therefore, we have that

$$\max_{Y_k\in\mathcal{Y}^+} l_k = \max\left\{q - e^{-\lambda\Delta}\underline{q},\ \left(1 - e^{-\lambda\Delta}\rho\right)\underline{q}\right\} \tag{35}$$

where $\underline{q} := \rho^{\lfloor \ln\frac{q}{\alpha_0}/\ln\rho\rfloor+1}\alpha_0$, which is the infimum of the quantization cell containing the upper bound q of I_k^-. For the case $Y_k \in \mathcal{Y}^-$, following the same discussion for $Y_k \in \mathcal{Y}^+$, we obtain

$$\max_{Y_k\in\mathcal{Y}^-} l_k = \max\left\{|p| - e^{-\lambda\Delta}\underline{p},\ \left(1 - e^{-\lambda\Delta}\rho\right)\underline{p}\right\}$$

where $\underline{p} := \rho^{\lfloor \ln\frac{|p|}{\alpha_0}/\ln\rho\rfloor+1}\alpha_0$. In view of (34), we have that the maximum of l_k over $Y_k \in \mathcal{Y}^- \cup \{0\} \cup \mathcal{Y}^+$ is equal to the right-hand side of (35).

2) $0 < p < q$: The estimation sets J_k constructed from Y_k which are compatible with 2) are given by

$$J_k = \left(e^{-\lambda\Delta}\rho^{i+1}\alpha_0, \rho^i\alpha_0\right], \left\lfloor \frac{\ln \frac{e^{\lambda\Delta}q}{\alpha_0}}{\ln\rho} \right\rfloor \leq i \leq \left\lfloor \frac{\ln \frac{p}{\alpha_0}}{\ln\rho} \right\rfloor, i\in\mathbb{Z}.$$

The maximum of l_k over Y_k can be obtained by following the discussion above for the case $Y_k \in \mathcal{Y}^+$ in 1). The only difference is that, in the current case, the lower bound p of I_k^- may be greater than that of J_k. This reasoning eventually leads to

$$\max_{Y_k} l_k = \begin{cases} \max\left\{q - e^{-\lambda\Delta}\underline{q}, \left(1 - e^{-\lambda\Delta}\rho\right)\underline{q}\right\} & \text{if } p \leq e^{-\lambda\Delta}\underline{q} \\ q - p & \text{if } p > e^{-\lambda\Delta}\underline{q} \end{cases} \tag{36}$$

where $\underline{q} := \rho^{\lfloor \ln\frac{q}{\alpha_0}/\ln\rho\rfloor+1}\alpha_0$, which is the same in (35). When I_k^- lies far from the origin, and thus, $p > e^{-\lambda\Delta}\underline{q}$, then there exists a case that the estimation set J_k becomes a superset of I_k^- for some Y_k.

As the second step of the proof, we evaluate the minimum of $\max_{Y_k} l_k$ obtained in (35) and (36) over the input $u(t)$. Note that $u(t)$ changes the position of I_k^- with respect to the origin. We first establish that $\min_{u(t)\text{ s.t. 2)}}\max_{Y_k} l_k \geq \min_{u(t)\text{ s.t. 1)}}\max_{Y_k} l_k$ as follows. Let us compare (36) with (35). If $p \leq e^{-\lambda\Delta}\underline{q}$, then $\max_{Y_k} l_k$ takes the same form for both cases 1) and 2), and that is increasing with q. Since q in the case 2) cannot be smaller than that in 1), the claim is true. Otherwise, if $p < e^{-\lambda\Delta}\underline{q}$, the right-hand side of (36) equals $q - p = e^{\lambda T}l_{k-1}$. This is greater than $\min_{u(t)\text{ s.t. 1)}}\max_{Y_k} l_k$ since it is possible to make $\min_{u(t)\text{ s.t. 1)}}\max_{Y_k} l_k$ less than or equal to $e^{\lambda T}l_{k-1}/2$ with the input such that $p = -q = e^{\lambda T}l_{k-1}/2$.

From the above discussion, we have that it suffices to consider the case 1) for the evaluation of $\min_{u(t)}\max_{x_k,\delta_k} l_k$ in (33). A simple calculation shows that the worst case length $\max_{Y_k} l_k$ in (35) is bounded from below and above by the following functions, both linear to q:

$$\frac{1 - \rho e^{-\lambda\Delta}}{1 + (1-\rho)e^{-\lambda\Delta}}q \leq \max\left\{q - e^{-\lambda\Delta}\underline{q},\ \left(1 - e^{-\lambda\Delta}\rho\right)\underline{q}\right\}$$
$$\leq \left(1 - \rho e^{-\lambda\Delta}\right)q. \tag{37}$$

Since both the lower bound and the upper bound in the above are increasing with q, the bounds are minimized when $q = -p = e^{\lambda T}l_{k-1}/2$, which means that I_k^- is symmetric about the origin. Substituting this into (37) leads to (33). ■

Proof of Theorem 3: (Sufficiency) If ρ satisfies (32), then there exists a logarithmic quantizer resulting in $(1 - \rho e^{-\lambda\Delta}) e^{\lambda T}/2 < 1$. Therefore, from Lemma 4, with the control input which places I_k^- symmetrically about the origin, we can make I_k narrower than the previous estimation set for any $x(0) \in I_0^-$ and $\delta_k \in [0,\Delta]$. Thus, repeating this procedure for each sampling period, we can achieve $l_k \to 0$ as $k \to \infty$, which implies the stability of the feedback system.

(Necessity) When $e^{\lambda T} \leq 2$, every $\rho \in (0,1)$ satisfies the sufficient condition (32) and hence the system is stabilizable. Consider the case of $e^{\lambda T} > 2$. Since I_k is a tight estimation set, when the system is stabilizable, the length l_k must converge to 0 for every $x(0)$ and $\{\delta_k\}_{k=0}^{\infty}$. From Lemma 4, it is required for the convergence of l_k that

$$\frac{1-\rho \mathrm{e}^{-\lambda\Delta}}{1+(1-\rho)\mathrm{e}^{-\lambda\Delta}}\frac{\mathrm{e}^{\lambda T}}{2} < 1.$$

The above inequality is equivalent to (31) when $\mathrm{e}^{\lambda T} > 2$. ■

B. Process with a Scalar-Valued State: Uniform Quantizer

In this section, we consider the infinite- and finite-range uniform quantizers that divide the state space uniformly. Given the width $w \in (0,\infty)$ of the quantization cells, the infinite-rage uniform quantizer is defined by

$$r(y) = [iw, (i+1)w) \tag{38}$$

where i is the integer such that $iw \le y < (i+1)w$.

Theorem 4: Let Assumptions 1, 2, and 4 hold. Consider the feedback system in Fig. 7 with the plant (10) and the infinite-range uniform quantizer (38). When $0 \le \Delta < \bar{\Delta}$, with $\bar{\Delta}$ given by (11), there exists a feedback controller such that, along solutions to the closed-loop, we have

$$\limsup_{t\to\infty} |x(t)| \le \begin{cases} 0 & \text{if } \mathrm{e}^{\lambda T} < 2 \\ \frac{1}{2}\frac{\mathrm{e}^{\lambda T} w}{1-(1-\mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}/2} & \text{if } \mathrm{e}^{\lambda T} \ge 2 \end{cases} \tag{39}$$

for every initial condition $x(0) \in \mathbb{R}$ and sampling offsets $\{\delta_k \in [0,\Delta]\}_{k=0}^{\infty}$.

Furthermore, if

$$\mathrm{e}^{\lambda T} \ge \frac{2}{1-\mathrm{e}^{-\lambda\Delta}\left(1-\mathrm{e}^{-\lambda\Delta}\right)} \tag{40}$$

then for any causal control law satisfying the setup in Section V, there exist an initial condition $x(0) \in \mathbb{R}$ and a sequence $\{\delta_k \in [0,\Delta]\}_{k=0}^{\infty}$ such that

$$\limsup_{t\to\infty} |x(t)| \ge \frac{1}{2}\frac{\mathrm{e}^{\lambda(T-\Delta)} w}{1-\left(1-\mathrm{e}^{-\lambda\Delta}\right)\mathrm{e}^{\lambda T}/2}. \tag{41}$$

Remark 4: We can see from (39) that for any $\Delta \in [0,\bar{\Delta})$, the uniform quantizer leads to a bounded solution, regardless of the width w of the quantizations, but with the caveat that the upper bound in the lower branch of (39) increases as both w and Δ increase. We also see from (39) that asymptotic stability is possible provided that $\mathrm{e}^{\lambda T} < 2$, but for larger values of $\mathrm{e}^{\lambda T}$ the state $x(t)$ may not converge to zero. In particular, when $\mathrm{e}^{\lambda T}$ exceeds the bound in (40), we can conclude from (41) that $x(t)$ will surely not converge to zero for some initial conditions and δ_k sequences.

Before we turn to the proof of Theorem 4, we study a more realistic scenario in which the quantizer's input range is a finite interval that does not cover the entire state space $\mathbb{R}$. Let us consider a finite-range quantizer that partitions $(-\sigma,\sigma)$, $\sigma > 0$, into an even number N cells of the same width $w = 2\sigma/N$ and

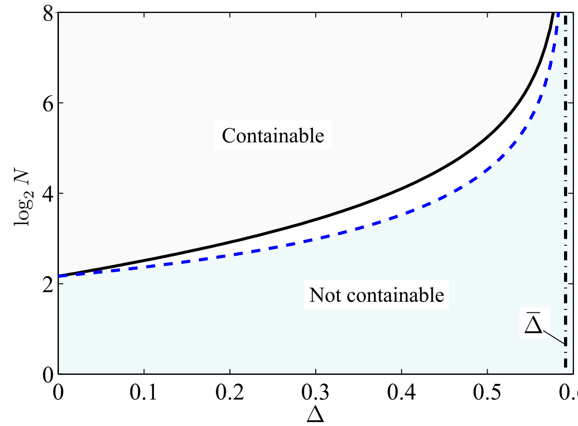

Fig. 9. Base-2 logarithms of the bounds of the number N of quantization cells versus the maximum clock offsets Δ: The sufficient condition in (43) (solid), the necessary condition in (44) (dashed), and the upper bound on the clock offset for stabilizability in (11) (dash–dot).

is defined by $Y_k = r(y_k)$, $\forall k \in \mathbb{Z}_+$, with

$$r(y) = \begin{cases} \left[\lfloor \frac{y}{w} \rfloor w, \lfloor \frac{y}{w}+1 \rfloor w\right) & \text{if } -\sigma+w \le y < \sigma \\ (-\sigma, -\sigma+w) & \text{if } -\sigma < y < -\sigma+w \\ \emptyset & \text{else.} \end{cases} \tag{42}$$

With the use of the finite-range quantizer (42), global stabilizability (as defined in Section IV) is not possible, so the control objective must be relaxed to containability, which has been studied in [36].

Definition 4: The feedback system in Fig. 7 with the N-level quantizer (42) is containable if for any sphere S centered at the origin, there exist an open neighborhood M of the origin and a feedback control law such that if $x(0) \in M$ then $x(t) \in S$ for all $t \ge 0$.

Theorem 5: Let Assumptions 1, 2, and 4 hold. Consider the feedback system in Fig. 7 with the plant (10) and the uniform quantizer (42). The feedback system is containable if $\mathrm{e}^{\lambda T} < 2$ or

$$0 \le \Delta < \bar{\Delta}, \quad N > \frac{\mathrm{e}^{\lambda T}}{1-(1-\mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}/2} \tag{43}$$

where $\bar{\Delta}$ is given in (11).

Conversely, if the systems is containable and (40) holds, then it follows that

$$0 \le \Delta < \bar{\Delta}, \quad N > \frac{\mathrm{e}^{\lambda(T-\Delta)}}{1-(1-\mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}/2}. \tag{44}$$

The following example illustrates this result.

Example 4: Consider the plant (10) with $\lambda = 1.0$ and fix the nominal sampling interval as $T = 1.5$. Fig. 9 shows the sufficient bound on N in (43) and the necessary bound in (44) for containability. The vertical axis is $\log_2 N$, which corresponds to the number of bits needed to send one of N symbols. From the figure, we see that for a clock offset larger than $\bar{\Delta}$ (i.e., on the right side of the dash–dot line), the bounds on N goes to infinity and the system is not containable.

Theorems 4 and 5 can be derived using the following lemma, which is the analogous of Lemma 4 in the logarithmic case. See the Appendix for the proofs of Lemma 5 and Theorem 4.

Lemma 5: Consider the feedback system in Fig. 7 with the scalar plant (10) and a uniform quantizer (42). For $k \in \mathbb{Z}_+$, suppose that the estimation set I_k^- satisfies

$$\left\{ \mathrm{e}^{\lambda t} x_k : x_k \in I_k^-, t \in [0, \Delta] \right\} \subset (-\sigma, \sigma) \tag{45}$$

i.e., I_k^- is contained in the quantizer's input during the kth sampling, and let l_k^- be the length of I_k^-. Then, the length l_k of I_k satisfies

$$\min_{\substack{u(t),\\ t\in((k-1)T+\Delta, kT)}} \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \le \kappa \mathrm{e}^{-\lambda T} l_k^- + \beta \tag{46}$$

where

$$\kappa := \left(1 - \mathrm{e}^{-\lambda \Delta}\right) \frac{\mathrm{e}^{\lambda T}}{2}, \quad \beta := \mathrm{e}^{-\lambda \Delta} w. \tag{47}$$

Furthermore, it holds that

$$\min_{\substack{u(t),\\ t\in((k-1)T+\Delta, kT)}} \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \ge \begin{cases} \frac{l_k^-}{2} & \text{if } l_k^- < 2w \\ \kappa \mathrm{e}^{-\lambda T} l_k^- + \mathrm{e}^{-\lambda \Delta} \beta & \text{else} \end{cases} \tag{48}$$

where the equality holds if $l_k^- < 2w$. In (46) and (48), the minimum occurs when I_k^- is placed symmetrically about the origin.

Proof of Theorem 5: (Sufficiency) We fix a specific value for $\sigma > 0$ and show that there exists an interval $I_0^- = [-l_0^-/2, l_0^-/2]$, $l_0^- > 0$, such that $x(t)$ can be contained in $(-\sigma, \sigma)$ for all $t \ge 0$. First, suppose that (43) holds. Then, the constant κ defined by (47) satisfies $\kappa < 1$ and there exists a constant l_0^- such that $w = 2\sigma/N$ and

$$\sigma > \frac{\mathrm{e}^{\lambda(T+\Delta)} \beta}{2(1-\kappa)} > \frac{\mathrm{e}^{\lambda \Delta}}{2} l_0^-. \tag{49}$$

With such a quantizer and I_0^-, we have that $|x(t)| < \mathrm{e}^{\lambda(T+\Delta)} \beta / \{2(1-\kappa)\}$ for $t \in [0, \Delta]$. Furthermore, we show below that the state can be bounded by this upper bound for all $t > \Delta$.

Notice that $I_0^- = [-l_0^-/2, l_0^-/2]$ satisfies (45) from (49) and is symmetric about the origin. Thus, from (46) and (49), we have that $l_0 < \beta/(1-\kappa)$. Therefore, in light of (46), $\kappa < 1$, and $l_k^- = \mathrm{e}^{\lambda T} l_{k-1}$, by a control input which places I_k^- symmetrically about the origin, we can make I_k so that $l_k < \beta/(1-\kappa)$ and

$$|x(kT + \Delta)| \le \frac{\mathrm{e}^{\lambda(T+\Delta)} l_{k-1}}{2} < \frac{\mathrm{e}^{\lambda(T+\Delta)} \beta}{2(1-\kappa)} \; \forall k \in \mathbb{Z}_+ \tag{50}$$

where $l_{-1} := \mathrm{e}^{-\lambda T} l_0^-$.

Moreover, we can bound $|x(t)|$ by the far right-hand side of (50) also for $t \in (kT + \Delta, (k+1)T + \Delta)$ as follows: From (50), there exists $t_1 \in (kT + \Delta, (k+1)T)$ such that $|x(t_1)| < \mathrm{e}^{\lambda(T+\Delta)} \beta / \{2(1-\kappa)\}$. Pick any $t_2 \in (t_1, (k+1)T)$ and consider the control input which takes the constant during $[t_1, t_2]$ to bring the center of the set $\{\mathrm{e}^{\lambda(t_1 - kT)} x_k : x_k \in I_k\}$ to the origin and zero for the rest. Then, since $|x((k+1)T + \Delta)|$ satisfies the bound in (50), $|x(t)|$ is also bounded by the same upper bound for $t > \Delta$.

On the other hand, when $\mathrm{e}^{\lambda T} < 2$ is satisfied, there exists I_0^- such that $l_0^- < 2w$. Then, from (48), $l_0 = l_0^-/2$, which results in $l_1^- = \mathrm{e}^{\lambda T} l_0 < l_0^-$ by the above control input, and this concludes containability for this case as well.

(Necessity) When the system is containable, there exists a control input so that the quantizer does not saturate, i.e., all $x_k \in I_k$ are bounded as $|x_k| < \sigma$ for all time k. Otherwise, we lose track of some states compatible with the past measurements. Thus, when (40) holds, the right-hand side of (41) in Theorem 4 must be smaller than 2σ from containability. This fact and $N = 2\sigma/w$ lead us to (44). ■

C. Process with a Vector-Valued State

For a specific class of general order systems, we can apply some of the results presented above for scalar systems. Consider the feedback system in Fig. 7 with a plant (1) for which the unstable part of A is real diagonalizable, i.e., A is similar to the matrix

$$\begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix}$$

where $A_u := \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{n_u})$, $\lambda_i > 0$, $i = 1, \ldots, n_u$, and $A_s \in \mathbb{R}^{(n-n_u)\times(n-n_u)}$ has no unstable eigenvalue. The stabilization of this system boils down to stabilizing the first-order systems $\dot{\xi}_i(t) = \lambda_i \xi_i(t) + \nu_i(t)$, $i = 1, \ldots, n_u$, based on the quantized values of $\xi_i(kT + \delta_k)$. If all the first-order systems are stabilizable with logarithmic quantizers (or containable with uniform quantizers), then the feedback system is stabilizable in the sense of Definition 2 (containable in the sense of Definition 4).

However, the results in Theorem 2 indicate that such conditions induced by the scalar case results may be conservative. The main difficulty in extending Theorem 2 to the quantization case lies in the evaluation of the volume of the estimation set I_k. Unlike in the no-quantization case, with quantization the observed state comes as a set in the state space and the set J_k becomes a band of exponential functions. Then, I_k is the intersection of exponential bands and obtaining a lower or upper bound of its volume analytically remains an open problem.

VII. Conclusion

We studied the stability of a linear system using an asynchronous sensor and controller connected with a communication channel. We first considered the quantization-free case and derived conditions on the clock offset for stabilizability. The condition for processes with a scalar-valued state gives a tight limitation on the offset, which depends on the level of instability of the plant and the sampling period. For processes with a vector-valued state, we showed that if the plant has at least two distinct poles, then the system is always stabilizable in finite time. We then study for the quantization case. For processes with a scalar-valued state with a logarithmic or uniform quantizer, we derived a necessary condition and a sufficient condition for stabilizability or containability, in terms of the coarseness of the quantizers.

Although a stabilizing algorithm has been given in the proofs, this algorithm is generally not suitable for practical controllers. The construction of practical control algorithms remains an important topic for future work. Other possible extensions include the case with disturbances, the output feedback case, and considering long delays which do not satisfy Assumption 2.

APPENDIX

We first present the proof of Lemma 5. Then, Theorem 4 is proved using Lemma 5.

Proof of Lemma 5: We follow the approach in the proof of Lemma 4 and evaluate the maximum of l_k over all possible Y_k. Let p and q denote the lower bound and the upper bound for I_k^-, respectively, i.e., $p := \inf_{\xi \in I_k^-} \xi$ and $q := \sup_{\xi \in I_k^-} \xi$, and assume that $|q| \geq |p|$.

The proof consists of three steps. First, we obtain an expression for $\max_{Y_k} l_k$ in terms of p and q. To do so, we consider the two cases 1) $p < 0 < q$ and 2) $0 \leq p < q$. We then establish that $\min_{u(t) \text{ s.t. } 1)} \max_{Y_k} l_k \leq \min_{u(t) \text{ s.t. } 2)} \max_{Y_k} l_k$. Finally, the inequalities (46) and (48) are proved using the bounds on $\min_{u(t)} \max_{Y_k} l_k$ obtained in the previous step.

Step 1: Consider the following two cases.

1) $p < 0 < q$: If in addition $q < w$, then $l_k = q$ or $l_k = |p|$. Thus, from the assumption $|q| \geq |p|$, we have

$$\max_{Y_k} l_k = q. \tag{51}$$

Otherwise, i.e., if $q \geq w$, we show that

$$\max_{Y_k} l_k = \max\left\{q - \mathrm{e}^{-\lambda\Delta}\underline{q},\ \left(1 - \mathrm{e}^{-\lambda\Delta}\right)\underline{q} + \mathrm{e}^{-\lambda\Delta} w\right\} \tag{52}$$

where $\underline{q}$ is the lower bound of the cell containing q, which is given by $\underline{q} := \lfloor q/w \rfloor w$. In the case of 1), the set of all possible Y_k is given by $\mathcal{Y}^+ \cup \mathcal{Y}^-$, where

$$\mathcal{Y}^+ := \left\{[iw, (i+1)w) : 0 \leq i \leq \left\lfloor \frac{\mathrm{e}^{\lambda\Delta} q}{w} \right\rfloor, \quad i \in \mathbb{Z}\right\}$$

$$\mathcal{Y}^- := \left\{[iw, (i+1)w) : \left\lfloor \frac{\mathrm{e}^{\lambda\Delta} p}{w} \right\rfloor \leq i \leq -1, \quad i \in \mathbb{Z}\right\}.$$

When $Y_k \in \mathcal{Y}^+$, it follows that

$$l_k = \min\{q, (i+1)w\} - \mathrm{e}^{-\lambda\Delta} iw,\ 0 \leq i \leq \left\lfloor \frac{\mathrm{e}^{\lambda\Delta} q}{w} \right\rfloor, \quad i \in \mathbb{Z}.$$

Regarding the above l_k, for Y_k such that $q < (i+1)w$, or equivalently $i \geq \lfloor q/w \rfloor$, we have $l_k = q - \mathrm{e}^{-\lambda\Delta} iw$. Since l_k is decreasing with i, it takes the maximum $q - \mathrm{e}^{-\lambda\Delta}\underline{q}$ when $i = \lfloor q/w \rfloor$; otherwise, i.e., if $i \leq \lfloor q/w \rfloor - 1$, then, $l_k = (i+1)w - \mathrm{e}^{-\lambda\Delta} iw$, and hence its maximum is $(1 - \mathrm{e}^{-\lambda\Delta})\underline{q} + \mathrm{e}^{-\lambda\Delta} w$. Therefore, (52) holds for $Y_k \in \mathcal{Y}^+$.

Following the same discussion, for $Y_k \in \mathcal{Y}^-$, we have that

$$\max_{Y_k} l_k = \max\left\{|p| - \mathrm{e}^{-\lambda\Delta}\left\lfloor \frac{|p|}{w} \right\rfloor w \quad \left(1 - \mathrm{e}^{-\lambda\Delta}\right)\left\lfloor \frac{|p|}{w} \right\rfloor w + \mathrm{e}^{-\lambda\Delta} w\right\}.$$

Noticing that $|q| \geq |p|$, this is smaller than or equal to the right-hand side of (52). Thus, (52) expresses the maximum over $\mathcal{Y}^+ \cup \mathcal{Y}^-$.

From (51) and (52), $\max_{Y_k} l_k$ with I_k^- satisfying 1) is given as

$$\max_{Y_k} l_k = \begin{cases} q - \mathrm{e}^{-\lambda\Delta}\underline{q} & \text{if } \underline{q} = 0 \text{ or } q - \underline{q} \geq \mathrm{e}^{-\lambda\Delta} w \\ \left(1 - \mathrm{e}^{-\lambda\Delta}\right)\underline{q} + \mathrm{e}^{-\lambda\Delta} w & \text{else.} \end{cases} \tag{53}$$

2) $0 \leq p < q$: In this case, the range of possible Y_k is given by

$$Y_k \in \left\{[iw, (i+1)w) : \left\lfloor \frac{p}{w} \right\rfloor \leq i \leq \left\lfloor \frac{\mathrm{e}^{\lambda\Delta} q}{w} \right\rfloor, i \in \mathbb{Z}\right\}.$$

Hence, it follows that

$$l_k = \min\{q, (i+1)w\} - \max\{p, \mathrm{e}^{-\lambda\Delta} iw\}$$
$$\left\lfloor \frac{p}{w} \right\rfloor \leq i \leq \left\lfloor \frac{\mathrm{e}^{\lambda\Delta} q}{w} \right\rfloor, \ i \in \mathbb{Z}.$$

Following a similar analysis to 1), we have that $\max_{Y_k} l_k$ with I_k^- satisfying 2) is given by

$$\max_{Y_k} l_k = \max\left\{q - \max\left\{p, \mathrm{e}^{-\lambda\Delta}\underline{q}\right\} \quad \underline{q} - \max\left\{p, \mathrm{e}^{-\lambda\Delta}\left(\underline{q} - w\right)\right\}\right\}. \tag{54}$$

Step 2: In this step, we show that $\min_{u(t) \text{ s.t. } 1)} \max_{Y_k} l_k \leq \min_{u(t) \text{ s.t. } 2)} \max_{Y_k} l_k$. From (53), the minimum of $\min_{u(t) \text{ s.t. } 1)} \max_{Y_k} l_k$ can be computed as

$$\min_{u(t) \text{ s.t. (i)}} \max_{Y_k} l_k = \begin{cases} L & \text{if } L < w \\ \max\left\{L - \mathrm{e}^{-\lambda\Delta}\lfloor \frac{L}{w} \rfloor w, \right. & \\ \left. \left(1 - \mathrm{e}^{-\lambda\Delta}\right)\lfloor \frac{L}{w} \rfloor w + \mathrm{e}^{-\lambda\Delta} w\right\} & \text{else} \end{cases} \tag{55}$$

where $L := l_k^-/2$. Meanwhile, it is difficult to obtain $\min_{u(t) \text{ s.t. } 2)} \max_{Y_k} l_k$ directly from (54). Thus, in what follows, we prove that for any I_k^- satisfying 2), $\max_{Y_k} l_k$ becomes greater than (55).

Suppose that I_k^- corresponds to the case 2) above and consider two cases (a) $L < w$ and (b) $L \geq w$.

(a) $L < w$: We prove that, for this case, it holds that $\max_{Y_k} l_k \geq L$, while $\min_{u(t) \text{ s.t. (i)}} \max_{Y_k} l_k = L$ from (55). To do so, three cases (a-1)–(a-3) are examined depending on the size of I_k^-:

(a-1) $p \geq \underline{q}$: In this case, I_k^- is contained in the quantization cell $\left[\underline{q}, \underline{q} + w\right)$. Thus, it is true that $\max_{Y_k} l_k = 2L \geq L$.

(a-2) $\underline{q} - w \leq p < \underline{q}$: From (54), we have that

$$\max_{Y_k} l_k = \max\left\{q - \max\left\{p, \mathrm{e}^{-\lambda\Delta}\underline{q}\right\},\ \underline{q} - p\right\}$$
$$\geq \max\left\{q - \underline{q},\ \underline{q} - p\right\} \tag{56}$$

where the inequality follows since $\underline{q} > p$ by (a-2) and $\underline{q} \geq \mathrm{e}^{-\lambda\Delta}\underline{q}$. The far right-hand side of (56) is greater than or equal to L since $q - \underline{q} < L$ implies that $\underline{q} - p = 2L - (q - \underline{q}) > L$.

(a-3) $p < \underline{q} - w$: From the condition, the quantization cell $\left[\underline{q} - w, \underline{q}\right)$ becomes a subset of I_k^-. Such Y_k results in I_k such that $l_k \geq w$, which is greater than L by the condition (a).

(b) $L \geq w$: To obtain a simpler expression to (54), consider two cases (b-1) and (b-2):

(b-1) $p \leq \mathrm{e}^{-\lambda\Delta}(\underline{q} - w)$: From (54), it follows that

$$\max_{Y_k} l_k = \max\left\{q - \mathrm{e}^{-\lambda\Delta}\underline{q},\ (1 - \mathrm{e}^{-\lambda\Delta})\underline{q} + \mathrm{e}^{-\lambda\Delta}w\right\}.$$

Noticing that $\underline{q} > 0$, we have that the above expression is the same as the right-hand side of (53). Since q and $\underline{q}$ cannot be smaller than or equal to those in the case 1), it holds that $\max_{Y_k} l_k > \min_{u(t) \text{ s.t. (i)}} \max_{Y_k} l_k$.

(b-2) $p > \mathrm{e}^{-\lambda\Delta}(\underline{q} - w)$: In this case, we have

$$\begin{aligned}\max_{Y_k} l_k &= \max\left\{q - \mathrm{e}^{-\lambda\Delta}\underline{q},\ \underline{q} - p\right\}\\ &\geq \underline{q} - p = 2L - \left(q - \underline{q}\right)\\ &> 2L - w \geq L.\end{aligned}$$

Here, the second inequality holds by the fact that $q - \underline{q} < w$ and the third inequality is obtained by the condition (b). On the other hand, from (55), it follows that $\min_{u(t) \text{ s.t. (i)}} \max_{Y_k} l_k < L$ and thus $\max_{Y_k} l_k > \min_{u(t) \text{ s.t. (i)}} \max_{Y_k} l_k$ for this case as well.

Step 3: From Step 2, we have that $\min_{u(t)} \max_{Y_k} l_k$ equals $\min_{u(t) \text{ s.t. 1)}} \max_{Y_k} l_k$ in (55) and the minimum occurs when $q = -p = L$. With κ and β in (47), the right-hand side of (55) is bounded from above as

$$\min_{\substack{u(t),\\ t\in((k-1)T+\Delta,kT)}} \max_{Y_k} l_k \leq \kappa \mathrm{e}^{-\lambda T} l_k^- + \beta$$

which concludes (46). Moreover, since (55) is bounded from below as

$$\min_{\substack{u(t),\\ t\in((k-1)T+\Delta,kT)}} \max_{Y_k} l_k \geq \begin{cases}\frac{l_k^-}{2} & \text{if } l_k^- < 2w\\ \kappa \mathrm{e}^{-\lambda T} l_k^- + \mathrm{e}^{-\lambda\Delta}\beta & \text{else}\end{cases}$$

we have (48). ∎

Now we are ready to prove Theorem 4.

Proof of Theorem 4 (Sufficiency) In the setup of Theorem 4, we employ the infinite-range uniform quantizer (38). Thus, (45) in Lemma 5 follows for all k. Since it follows that $\kappa < 1$ by $\Delta < \bar{\Delta}$ and from (46), we can make I_k so that for any $x(0) \in I_0^-$ and $\{\delta_k \in [0, \Delta]\}_{k=0}^{\infty}$

$$\limsup_{k\to\infty} l_k \leq \frac{\beta}{1-\kappa}$$

by placing I_k^- symmetrically about the origin. Thus, using the control input stated in the proof of Theorem 5, we have

$$\limsup_{k\to\infty} |x(t)| \leq \frac{\mathrm{e}^{\lambda(T+\Delta)}}{2}\frac{\beta}{1-\kappa}$$

which establishes the lower branch of (39). Furthermore, if $\mathrm{e}^{\lambda T} < 2$, then it follows that $\beta/(1-\kappa) < 2\mathrm{e}^{-\lambda T}w$. Hence, there exists $k' \in \mathbb{Z}_+$ such that $l_{k'} < 2\mathrm{e}^{-\lambda T}w$. By applying the above stated input, we obtain from (48) that $l_{k'+1} = \mathrm{e}^{\lambda T} l_{k'}/2$, which implies $\lim_{k\to\infty} l_k = 0$ and thus we have (39).

(Necessity) If (40) holds, then $\mathrm{e}^{-\lambda\Delta}\beta/(1-\kappa) \geq 2\mathrm{e}^{-\lambda T}w$. Thus, by (48), there exists a possible path of the state resulting in

$$\limsup_{k\to\infty} l_k \geq \frac{\mathrm{e}^{-\lambda\Delta}\beta}{1-\kappa}.$$

The inequality (41) follows from this. ∎

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