Lyapunov small-gain theorems
for not necessarily ISS hybrid systems

Andrii Mironchenko, Guosong Yang and Daniel Liberzon

Institute of Mathematics
University of Würzburg

Coordinated Science Laboratory
University of Illinois at Urbana-Champaign

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Class of systems

\[ \dot{x} \in f(x, u), \quad (x, u) \in C, \]
\[ x^+ \in g(x, u), \quad (x, u) \in D. \]

(\Sigma)

- \( E \subset \mathbb{R}_+ \times \mathbb{N} \) is a compact hybrid time domain if

\[ E = \bigcup_{j=0}^J ([t_j, t_{j+1}], j) \]

for some \( 0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1} \).

- \( E \subset \mathbb{R}_+ \times \mathbb{N} \) is a hybrid time domain if \( \forall (T, J) \in E, \)
  \( E \cap ([0, T] \times \{0, 1, \ldots, J\}) \) is a compact hybrid time domain.
Comparison functions

\[ K_\infty := \{ \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \gamma(0) = 0, \ \gamma \text{ is continuous, increasing and unbounded} \} \]

\[ \mathcal{L} := \{ \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly decreasing and } \lim_{t \to \infty} \gamma(t) = 0 \} \]

\[ \mathcal{KL} := \{ \beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \beta(\cdot, t) \in K, \ \forall t \geq 0, \ \beta(r, \cdot) \in \mathcal{L}, \ \forall r > 0 \} \]
Definition (ISS)

A set of solution pairs $S$ is pre-ISS w.r.t. $A$ if

\[ \forall (x, u) \in S, \forall (t, j) \in \text{dom } x \]

\[ |x(t, j)|_A \leq \max \{ \beta(|x(0, 0)|_A, t + j), \gamma(\|u\|_{(t, j)}) \} . \]

- $\Sigma$ is pre-ISS w.r.t. $A$ if $S = \{ \text{all solution pairs } (x, u) \text{ of } \Sigma \}$ is pre-ISS w.r.t. $A$.
- $\Sigma$ is ISS w.r.t. $A$ if $\Sigma$ is pre-ISS w.r.t. $A$ and all solution pairs are complete.
**Definition**

$V$ is **exponential ISS-LF w.r.t.** $\mathcal{A} \subset X$ \iff $\exists \psi_1, \psi_2 \in K_\infty$, $c, d \in \mathbb{R}$:

- $\psi_1(|x|_A) \leq V(x) \leq \psi_2(|x|_A) \quad \forall x \in X$

- $V(x) \geq \chi(|u|) \Rightarrow \begin{cases} 
\dot{V}(x; y) \leq -cV(x) & \forall (x, u) \in C, y \in f(x, u), \\
V(y) \leq e^{-d}V(x) & \forall (x, u) \in D, y \in g(x, u).
\end{cases}$
Lyapunov functions

Definition

\( V \) is exponential ISS-LF w.r.t. \( A \subset X \) : \iff \exists \psi_1, \psi_2 \in \mathcal{K}_\infty, c, d \in \mathbb{R}:

- \( \psi_1(|x|_A) \leq V(x) \leq \psi_2(|x|_A) \quad \forall \ x \in X \)

- \( V(x) \geq \chi(|u|) \Rightarrow \left\{ \begin{array}{l}
\dot{V}(x; y) \leq -cV(x) \quad \forall \ (x, u) \in C, y \in f(x, u), \\
V(y) \leq e^{-d}V(x) \quad \forall \ (x, u) \in D, y \in g(x, u).
\end{array} \right. \)

Proposition

Let \( V \) be exp. ISS-LF for \( \Sigma \) w.r.t. \( A \) with \( d \neq 0 \). \( \forall \mu \geq 1, \forall \eta, \lambda > 0, \)
\( S[\eta, \lambda, \mu] := \text{set of solution pairs } (x, u) \text{ satisfying} \)

\[-(d - \eta)(j - i) - (c - \lambda)(t - s) \leq \mu \quad \forall \ (t, j), (s, i) \in \text{dom } x.\]

Then \( S[\eta, \lambda, \mu] \) is pre-ISS w.r.t. \( A \).
Interconnected systems

\[ \Sigma : \begin{cases} \Sigma_i : x_i \in f_i(x, u), & (x, u) \in C, \\ x_i^+ \in g_i(x, u), & (x, u) \in D, \\ i = 1, \ldots, n \end{cases} \]

**Definition**

\( V_i : X_i \rightarrow \mathbb{R}_+ \) is exponential ISS LF for \( \Sigma_i \) w.r.t. \( A_i \subset X_i \) :

1) \( \exists \chi_{ij}, \chi_i \in \mathcal{K}, c_i, d_i \in \mathbb{R} : \forall (x, u) \in C, \forall y_i \in f_i(x, u), \)

\[ V_i(x_i) \geq \max \left\{ \max_{j=1,j\neq i} \chi_{ij}(V_j(x_j)), \chi_i(|u|) \right\} \Rightarrow \dot{V}_i(x_i; y_i) \leq -c_i(V_i(x_i)). \]

2) \( \exists d_i \in \mathbb{R} : \forall (x, u) \in D, \forall y_i \in g_i(x, u) \)

\[ V_i(y_i) \leq \max \left\{ e^{-d_i} V_i(x_i), \max_{j=1,j\neq i} \chi_{ij}(V_j(x_j)), \chi_i(|u|) \right\} . \]

D. Liberzon, D. Nesic, A. Teel. Small-gain theorems of LaSalle type for hybrid systems, CDC 2012.

- Small-gain theorems for interconnections of 2 ISS systems
- Modification method for interconnections with not necessarily ISS subsystems.
- LaSalle-Krasovskii type theorem for hybrid systems
Literature overview

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- Small-gain theorems for interconnections of $n$ impulsive systems with matched instabilities.
- Stability conditions for nonexponential Lyapunov functions

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Interconnections of 2 systems

ISS-LF for $\Sigma_i$

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov functions for $\Sigma_i, i = 1, 2$ iff

- $\forall (x, u) \in C, \forall y_1 \in f_1(x, u)$
  \[ V_1(x_1) \geq \max \{ \chi_{12}(V_2(x_2)), \chi_1(|u|) \} \Rightarrow \dot{V}_1(x_1; y_1) \leq -c_1 V_1(x_1), \]

- $\forall (x, u) \in D, \forall y_1 \in g_1(x, u)$
  \[ V_1(y_1) \leq \max \{ e^{-d_1} V_1(x_1), \chi_{12}(V_2(x_2)), \chi_1(|u|) \}. \]

- $\forall (x, u) \in C, \forall y_2 \in f_2(x, u)$
  \[ V_2(x_2) \geq \max \{ \chi_{21}(V_1(x_1)), \chi_2(|u|) \} \Rightarrow \dot{V}_2(x_2) \leq -c_2 V_2(x_2), \]

- $\forall (x, u) \in D, \forall y_2 \in g_2(x, u)$
  \[ V_2(y_2) \leq \max \{ e^{-d_2} V_2(x_2), \chi_{21}(V_1(x_1)), \chi_2(|u|) \}. \]
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- $\forall (x, u) \in D, \forall y_1 \in g_1(x, u)$
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How to use these two LFs to study ISS of the interconnection?
Interconnections of 2 systems

ISS-LF for $\Sigma_i$

$V_i : X_i \rightarrow \mathbb{R}_+$ is ISS-Lyapunov functions for $\Sigma_i$, $i = 1, 2$ iff

1. $\forall (x, u) \in C, \forall y_1 \in f_1(x, u)$
   
   $V_1(x_1) \geq \max \{\chi_{12}(V_2(x_2)), \chi_1(|u|)\} \Rightarrow \dot{V}_1(x_1; y_1) \leq -c_1 V_1(x_1),$

2. $\forall (x, u) \in D, \forall y_1 \in g_1(x, u)$
   
   $V_1(y_1) \leq \max \{e^{-d_1} V_1(x_1), \chi_{12}(V_2(x_2)), \chi_1(|u|)\}.$

3. $\forall (x, u) \in C, \forall y_2 \in f_2(x, u)$
   
   $V_2(x_2) \geq \max \{\chi_{21}(V_1(x_1)), \chi_2(|u|)\} \Rightarrow \dot{V}_2(x_2) \leq -c_2 V_2(x_2),$

4. $\forall (x, u) \in D, \forall y_2 \in g_2(x, u)$
   
   $V_2(y_2) \leq \max \{e^{-d_2} V_2(x_2), \chi_{21}(V_1(x_1)), \chi_2(|u|)\}.$

This depends on $\chi_{12}, \chi_{21}$ and coefficients $c_i, d_i$!
Theorem (Liberzon, Nesic, Teel, IEEE TAC, 2014)

Let $V_1, V_2$ be ISS-Lyapunov function for $\Sigma_1, \Sigma_2$ with gains $\chi_{12}, \chi_{21}$. Let also $c_1, c_2, d_1, d_2 > 0$. Then

$$\chi_{12} \circ \chi_{21} < id$$

(SGC)

$\Rightarrow$

- $\Sigma$ is ISS.
- $V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is an ISS Lyapunov function for $\Sigma$. 

The case when $c_i > 0, d_i > 0, i = 1, 2$
The case when $c_i > 0$, $d_i > 0$, $i = 1, 2$

**Theorem (Liberzon, Nesic, Teel, IEEE TAC, 2014)**

Let $V_1$, $V_2$ be ISS-Lyapunov function for $\Sigma_1$, $\Sigma_2$ with gains $\chi_{12}$, $\chi_{21}$. Let also $c_1, c_2, d_1, d_2 > 0$. Then

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$\implies$

- $\Sigma$ is ISS.
- $V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is an ISS Lyapunov function for $\Sigma$.

If $\chi_{ij}$ are linear, $\rho$ can be chosen linear and s.t. $V$ is an exponential LF.
The case when $c_i > 0$, $d_i > 0$, $i = 1, 2$

Theorem (Liberzon, Nesic, Teel, IEEE TAC, 2014)

Let $V_1, V_2$ be ISS-Lyapunov function for $\Sigma_1, \Sigma_2$ with gains $\chi_{12}, \chi_{21}$. Let also $c_1, c_2, d_1, d_2 > 0$. Then

$$\chi_{12} \circ \chi_{21} < \text{id}$$

(SGC)

$\Rightarrow$

- $\Sigma$ is ISS.
- $V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is an ISS Lyapunov function for $\Sigma$.

If $\chi_{ij}$ are linear, $\rho$ can be chosen linear and s.t. $V$ is an exponential LF.

What to do if some of $c_i, d_i$ are $< 0$?
Modification of "bad" subsystems when $\chi_{ij}$ are linear


\[ V_i = LF \text{ for } \Sigma_i \text{ gains } \chi_{ij} \]
\[ W_i = LF \text{ for } \tilde{\Sigma}_i \text{ gains } \tilde{\chi}_{ij} \]

Restrict jumping rate of $\Sigma_i$
\[ \forall i : c_i < 0 \text{ and } \forall i : d_i < 0 \]

The gains increase exponentially with chatter bound!

SGT (for systems with ISS subsystems)

\[ W = LF \text{ for } \tilde{\Sigma} \]

Conditions for ISS of $\Sigma$
A new small-gain theorem

Define $\Gamma_M := (\chi_{ij})_{n \times n}$.

**Theorem**

Let $V_i$ be exp. ISS LF for $\Sigma_i$ w.r.t. $A_i$ with $d_i \neq 0$ and linear gains $\chi_{ij}$.

$$\rho(\Gamma_M) < 1 \implies V(x) := \max_{i=1}^{n} \frac{1}{s_i} V_i(x_i)$$

is an exp. ISS LF for $\Sigma$ w.r.t. $A = A_1 \times \ldots \times A_n$ with rate coefficients

$$c := \min_{i=1}^{n} c_i, \quad d := \min_{i,j:j \neq i} \left\{ d_i, - \ln \left( \frac{s_j}{s_i} \chi_{ij} \right) \right\}.$$ 

This LF can be used to prove ISS if all $c_i > 0$ or all $d_i > 0$. 
Modification of "bad" subsystems when $\chi_{ij}$ are linear

Improved modification method

\begin{align*}
V_i &= \text{LF for } \Sigma_i \\
W_i &= \text{LF for } \tilde{\Sigma}_i \\
W &= \text{LF for } \tilde{\Sigma}
\end{align*}

Restrict jumping rate of $\Sigma_i$

$\forall i : c_i < 0 \text{ or } \forall i : d_i < 0$

The gains increase exponentially with chatter bound!

SGT (for systems with matched instabilities)

Conditions for ISS of $\Sigma$
Making discrete dynamics ISS

- Define \( I_d := \{i \in \{1, \ldots, n\} : d_i < 0\} \).
- For all \( i \in I_d \) restrict the frequency of jumps of \( \Sigma_i \) by
  \[
  j - k \leq \delta_i(t - s) + N^i_0,
  \]
  (ADT)
  where \( \delta_i, N^i_0 > 0 \) and \( (t, j), (s, k) \in \text{dom } x \).
- It can be modeled by the clock
  \[
  \begin{align*}
  \dot{\tau}_i & \in [0, \delta_i], \quad \tau_i \in [0, N^i_0], \\
  \tau^+_i & = \tau_i - 1, \quad \tau_i \in [1, N^i_0].
  \end{align*}
  \]
- LF for \( \tilde{\Sigma}_i \):
  \[
  W_i(z_i) := \begin{cases} 
  V_i(x_i), & i \notin I_d, \\
  e^{L_i\tau_i} V_i(x_i), & i \in I_d.
  \end{cases}
  \]
Proposition: ISS LF for modified subsystems

1) \( \forall (z, u) \in \tilde{C} \) and \( \forall y_i \in \tilde{f}_i(z, u) \),

\[
W_i(z_i) \geq \max \left\{ \max_{j=1}^n \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u| \right\} \Rightarrow \dot{W}_i(z_i; y_i) \leq -\tilde{c}_i W_i(z_i),
\]

where \( \tilde{c}_i = c_i \) for \( i \notin I_d \); \( \tilde{c}_i = c_i - L_i \delta_i \) for \( i \in I_d \) and

\[
\tilde{\chi}_i := \chi_i, \quad \tilde{\chi}_{ij} := \chi_{ij}, \quad i \notin I_d,
\]

\[
\tilde{\chi}_i := e^{L_i N_0^i} \chi_i, \quad \tilde{\chi}_{ij} := e^{L_i N_0^i} \chi_{ij}, \quad i \in I_d.
\]

2) \( \forall (z, u) \in \tilde{D} \) and \( \forall y_i \in \tilde{g}_i(z, u) \),

\[
W_i(y_i) \leq \max \left\{ e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^n \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u| \right\},
\]

where \( \tilde{d}_i = d_i \) for \( i \notin I_d \) and \( \tilde{d}_i = d_i + L_i \) for \( i \in I_d \).
Example

Let $c_1 > 0$, $d_1 < 0$, $c_2 > 0$ and $d_2 < 0$.

- Instabilities are matched $\Rightarrow$ no modification.
- SGT $\Rightarrow$ LF $V$ for interconnection with $c > 0$ and $d < 0$.

Let $c_1 > 0$, $d_1 < 0$, $c_2 < 0$ and $d_2 > 0$.

- Instabilities are not matched $\Rightarrow$ modify $\Sigma_2$.

\[
\bar{\Gamma} = \begin{bmatrix}
0 & \tilde{\chi}_{12} \\
\tilde{\chi}_{21} & 0
\end{bmatrix} = \begin{bmatrix}
0 & e^{L_1 N_0^1} \chi_{12} \\
\chi_{21} & 0
\end{bmatrix}.
\]

- SGT $\Rightarrow$

\[
\chi_{12}\chi_{21} < e^{-L_1 N_0^1}.
\]

- Since $L_1 = -d_1 + \varepsilon$; $N_0^1 \geq 1$ $\Rightarrow$

\[
\chi_{12}\chi_{21} \leq e^{d_1}.
\]
Summary and Outlook

Main results
- New small-gain theorem for hybrid systems.
- If instabilities are matched $\Rightarrow$ no modification is needed.
- Less restrictive modification method

Outlook

Thank you for attention!

A. Mironchenko, G. Yang, D. Liberzon
### Main results

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- If instabilities are matched $\Rightarrow$ no modification is needed.
- Less restrictive modification method

### Outlook

- Can one avoid modifications if the instabilities are not matched?
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Outlook

- Can one avoid modifications if the instabilities are not matched?

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