Lyapunov small-gain theorems for not necessarily ISS hybrid systems

Andrii Mironchenko, Guosong Yang and Daniel Liberzon

Institute of Mathematics
University of Würzburg

Coordinated Science Laboratory
University of Illinois at Urbana-Champaign

MTNS 2014
Groningen, 9 July 2014

Class of systems

$$\begin{aligned} \dot{x} &\in f(x,u), \quad (x,u) \in C, \\ x^+ &\in g(x,u), \quad (x,u) \in D. \end{aligned} \qquad (\Sigma)$$

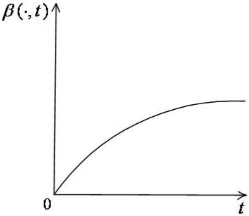
- $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a compact hybrid time domain if

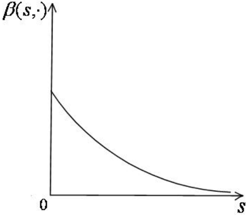
$$E = \bigcup_{j=0}^{J} ([t_j, t_{j+1}], j)$$

  for some $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$.
- $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a hybrid time domain if $\forall (T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain.

Comparison functions

$$\mathcal{K}_\infty := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma(0) = 0,\ \gamma \text{ is continuous, increasing and unbounded}\}$$
$$\mathcal{L} := \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly decreasing and } \lim_{t\to\infty} \gamma(t) = 0\right\}$$
$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid\ \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}$$





Input-to-state stability

Definition (ISS)

A set of solution pairs $\mathcal{S}$ is pre-ISS w.r.t. $\mathcal{A}$ $\quad :\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$: $\forall (x, u) \in \mathcal{S}$, $\forall (t, j) \in \operatorname{dom} x$

$$|x(t,j)|_{\mathcal{A}} \leq \max\left\{\beta(|x(0,0)|_{\mathcal{A}}, t+j), \gamma(\|u\|_{(t,j)})\right\}.$$

- Σ is pre-ISS w.r.t. $\mathcal{A}$ if $\mathcal{S} = \{$ all solution pairs (x, u) of $\Sigma\}$ is pre-ISS w.r.t. $\mathcal{A}$.
- Σ is ISS w.r.t. $\mathcal{A}$ if Σ is pre-ISS w.r.t. $\mathcal{A}$ and all solution pairs are complete.

Lyapunov functions

Definition

V is exponential ISS-LF w.r.t. $\mathcal{A} \subset X$ $:\Leftrightarrow$ $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$, $c, d \in \mathbb{R}$:

- $\psi_1(|x|_{\mathcal{A}}) \leq V(x) \leq \psi_2(|x|_{\mathcal{A}}) \qquad \forall\, x \in X$
- $V(x) \geq \chi(|u|) \Rightarrow \begin{cases} \dot{V}(x;y) \leq -cV(x) & \forall\,(x,u) \in C, y \in f(x,u), \\ V(y) \leq e^{-d} V(x) & \forall\,(x,u) \in D, y \in g(x,u). \end{cases}$

Lyapunov functions

Definition

V is exponential ISS-LF w.r.t. $\mathcal{A} \subset X$ $:\Leftrightarrow$ $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$, $c, d \in \mathbb{R}$:

- $\psi_1(|x|_\mathcal{A}) \leq V(x) \leq \psi_2(|x|_\mathcal{A}) \qquad \forall\, x \in X$
- $V(x) \geq \chi(|u|) \Rightarrow \begin{cases} \dot{V}(x; y) \leq -cV(x) & \forall\, (x,u) \in C, y \in f(x,u), \\ V(y) \leq e^{-d} V(x) & \forall\, (x,u) \in D, y \in g(x,u). \end{cases}$

Proposition

Let V be exp. ISS-LF for Σ w.r.t. $\mathcal{A}$ with $d \neq 0$. $\forall \mu \geq 1, \forall \eta, \lambda > 0$,
$\mathcal{S}[\eta, \lambda, \mu]$:= set of solution pairs (x, u) satisfying

$$-(d-\eta)(j-i) - (c-\lambda)(t-s) \leq \mu \quad \forall\, (t,j), (s,i) \in \operatorname{dom} x.$$

Then $\mathcal{S}[\eta, \lambda, \mu]$ is pre-ISS w.r.t. $\mathcal{A}$.

Interconnected systems

$$\Sigma : \begin{cases} \Sigma_i : \begin{array}{rl} \dot{x}_i \in f_i(x,u), & (x,u) \in C, \\ x_i^+ \in g_i(x,u), & (x,u) \in D, \end{array} \\ i = 1, \dots, n \end{cases}$$

Definition

$V_i : X_i \to \mathbb{R}_+$ is exponential ISS LF for Σ_i w.r.t. $\mathcal{A}_i \subset X_i$ $\quad :\Leftrightarrow$

1) $\exists \chi_{ij}, \chi_i \in \mathcal{K}$, $c_i, d_i \in \mathbb{R}$: $\quad \forall (x,u) \in C$, $\forall y_i \in f_i(x,u)$,

$$V_i(x_i) \geq \max\left\{ \max_{j=1, j\neq i}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|) \right\} \Rightarrow \dot{V}_i(x_i; y_i) \leq -c_i\,(V_i(x_i)).$$

2) $\exists d_i \in \mathbb{R}$: $\quad \forall (x,u) \in D$, $\forall y_i \in g_i(x,u)$

$$V_i(y_i) \leq \max\left\{ e^{-d_i} V_i(x_i), \max_{j=1, j\neq i}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|) \right\}.$$

Literature overview

- D. Liberzon, D. Nesic, A. Teel. Lyapunov-Based Small-Gain Theorems for Hybrid Systems, IEEE TAC, 2014.
- D. Liberzon, D. Nesic, A. Teel. Small-gain theorems of LaSalle type for hybrid systems, CDC 2012.
- Small-gain theorems for interconnections of 2 ISS systems
- Modification method for interconnections with not necessarily ISS subsystems.
- LaSalle-Krasovskii type theorem for hybrid systems

Literature overview

- D. Liberzon, D. Nesic, A. Teel. Lyapunov-Based Small-Gain Theorems for Hybrid Systems, IEEE TAC, 2014.
- D. Liberzon, D. Nesic, A. Teel. Small-gain theorems of LaSalle type for hybrid systems, CDC 2012.
- Small-gain theorems for interconnections of 2 ISS systems
- Modification method for interconnections with not necessarily ISS subsystems.
- LaSalle-Krasovskii type theorem for hybrid systems

- S. Dashkovskiy, A. Mironchenko. Input-to-State Stability of Nonlinear Impulsive Systems, SICON, 2013.
- S. Dashkovskiy, M. Kosmykov, A. Mironchenko, L. Naujok. Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods, Nonlinear Analysis: Hybrid Systems, 2012.
- Small-gain theorems for interconnections of n impulsive systems with matched instabilities.
- Stability conditions for nonexponential Lyapunov functions

Literature overview

- D. Liberzon, D. Nesic, A. Teel. Lyapunov-Based Small-Gain Theorems for Hybrid Systems, IEEE TAC, 2014.
- D. Liberzon, D. Nesic, A. Teel. Small-gain theorems of LaSalle type for hybrid systems, CDC 2012.
- Small-gain theorems for interconnections of 2 ISS systems
- Modification method for interconnections with not necessarily ISS subsystems.
- LaSalle-Krasovskii type theorem for hybrid systems

- S. Dashkovskiy, A. Mironchenko. Input-to-State Stability of Nonlinear Impulsive Systems, SICON, 2013.
- S. Dashkovskiy, M. Kosmykov, A. Mironchenko, L. Naujok. Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods, Nonlinear Analysis: Hybrid Systems, 2012.
- Small-gain theorems for interconnections of n impulsive systems with matched instabilities.
- Stability conditions for nonexponential Lyapunov functions

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\forall (x,u) \in C$, $\forall y_1 \in f_1(x,u)$
  $V_1(x_1) \geq \max\{\chi_{12}(V_2(x_2)), \chi_1(|u|_U)\} \;\Rightarrow\; \dot V_1(x_1; y_1) \leq -c_1 V_1(x_1)$,
- $\forall (x,u) \in D$, $\forall y_1 \in g_1(x,u)$
  $V_1(y_1) \leq \max\{e^{-d_1} V_1(x_1), \chi_{12}(V_2(x_2)), \chi_1(|u|)\}$.

- $\forall (x,u) \in C$, $\forall y_2 \in f_2(x,u)$
  $V_2(x_2) \geq \max\{\chi_{21}(V_1(x_1)), \chi_2(|u|_U)\} \;\Rightarrow\; \dot V_2(x_2) \leq -c_2 V_2(x_2)$,
- $\forall (x,u) \in D$, $\forall y_2 \in g_2(x,u)$
  $V_2(y_2) \leq \max\{e^{-d_2} V_2(x_2), \chi_{21}(V_1(x_1)), \chi_2(|u|)\}$.

Interconnections of 2 systems

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\forall (x,u) \in C$, $\forall y_1 \in f_1(x,u)$
  $V_1(x_1) \geq \max\left\{\chi_{12}(V_2(x_2)), \chi_1(|u|_U)\right\} \;\Rightarrow\; \dot{V}_1(x_1; y_1) \leq -c_1 V_1(x_1)$,
- $\forall (x,u) \in D$, $\forall y_1 \in g_1(x,u)$
  $V_1(y_1) \leq \max\left\{e^{-d_1} V_1(x_1), \chi_{12}(V_2(x_2)), \chi_1(|u|)\right\}$.

- $\forall (x,u) \in C$, $\forall y_2 \in f_2(x,u)$
  $V_2(x_2) \geq \max\left\{\chi_{21}(V_1(x_1)), \chi_2(|u|_U)\right\} \;\Rightarrow\; \dot{V}_2(x_2) \leq -c_2 V_2(x_2)$,
- $\forall (x,u) \in D$, $\forall y_2 \in g_2(x,u)$
  $V_2(y_2) \leq \max\left\{e^{-d_2} V_2(x_2), \chi_{21}(V_1(x_1)), \chi_2(|u|)\right\}$.

How to use these two LFs to study ISS of the interconnection?

Interconnections of 2 systems

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\forall (x,u) \in C$, $\forall y_1 \in f_1(x,u)$
  $V_1(x_1) \geq \max\left\{\chi_{12}(V_2(x_2)), \chi_1(|u|_U)\right\} \ \Rightarrow \ \dot{V}_1(x_1; y_1) \leq -c_1 V_1(x_1),$
- $\forall (x,u) \in D$, $\forall y_1 \in g_1(x,u)$
  $V_1(y_1) \leq \max\left\{e^{-d_1} V_1(x_1), \chi_{12}(V_2(x_2)), \chi_1(|u|)\right\}.$

- $\forall (x,u) \in C$, $\forall y_2 \in f_2(x,u)$
  $V_2(x_2) \geq \max\left\{\chi_{21}(V_1(x_1)), \chi_2(|u|_U)\right\} \ \Rightarrow \ \dot{V}_2(x_2) \leq -c_2 V_2(x_2),$
- $\forall (x,u) \in D$, $\forall y_2 \in g_2(x,u)$
  $V_2(y_2) \leq \max\left\{e^{-d_2} V_2(x_2), \chi_{21}(V_1(x_1)), \chi_2(|u|)\right\}.$

This depends on χ_{12}, χ_{21} and coefficients c_i, d_i!

The case when $c_i > 0, d_i > 0, i = 1, 2$

Theorem (Liberzon, Nesic, Teel, IEEE TAC, 2014)

Let V_1, V_2 be ISS-Lyapunov function for Σ_1, Σ_2 with gains χ_{12}, χ_{21}. Let also $c_1, c_2, d_1, d_2 > 0$. Then

$$\chi_{12} \circ \chi_{21} < id \quad \text{(SGC)}$$

$\Rightarrow$

- *Σ is ISS.*
- *$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is an ISS Lyapunov function for Σ.*

The case when $c_i > 0, d_i > 0, i = 1, 2$

Theorem (Liberzon, Nesic, Teel, IEEE TAC, 2014)

Let V_1, V_2 be ISS-Lyapunov function for Σ_1, Σ_2 with gains χ_{12}, χ_{21}. Let also $c_1, c_2, d_1, d_2 > 0$. Then

$$\chi_{12} \circ \chi_{21} < id \qquad \text{(SGC)}$$

$\Rightarrow$

- *Σ is ISS.*
- *$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is an ISS Lyapunov function for Σ.*

If χ_{ij} are linear, ρ can be chosen linear and s.t. V is an exponential LF.

The case when $c_i > 0, d_i > 0, i = 1, 2$

Theorem (Liberzon, Nesic, Teel, IEEE TAC, 2014)

Let V_1, V_2 be ISS-Lyapunov function for Σ_1, Σ_2 with gains χ_{12}, χ_{21}. Let also $c_1, c_2, d_1, d_2 > 0$. Then

$$\chi_{12} \circ \chi_{21} < id \qquad \text{(SGC)}$$

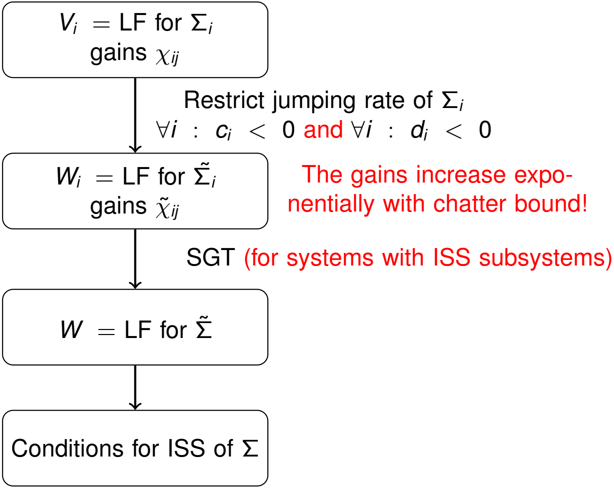
$\Rightarrow$

- *Σ is ISS.*
- *$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is an ISS Lyapunov function for Σ.*

If χ_{ij} are linear, ρ can be chosen linear and s.t. V is an exponential LF.

What to do if some of c_i, d_i are < 0?

Method due to Liberzon, Nesic, Teel, IEEE TAC, 2014.



A new small-gain theorem

Define $\Gamma_M := (\chi_{ij})_{n \times n}$.

Theorem

Let V_i be exp. ISS LF for Σ_i w.r.t. $\mathcal{A}_i$ with $d_i \neq 0$ and linear gains χ_{ij}.

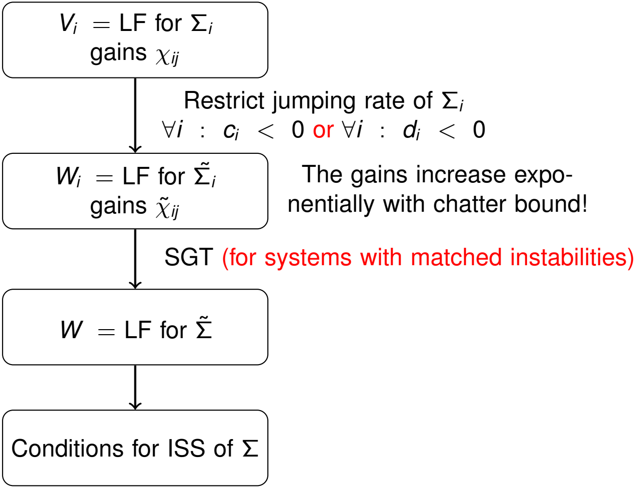
$$\rho(\Gamma_M) < 1 \quad \Rightarrow \quad V(x) := \max_{i=1}^{n} \frac{1}{s_i} V_i(x_i)$$

is an exp. ISS LF for Σ w.r.t. $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ with rate coefficients

$$c := \min_{i=1}^{n} c_i, \quad d := \min_{i,j:j \neq i} \left\{ d_i, -\ln\left(\frac{s_j}{s_i}\chi_{ij}\right) \right\}.$$

This LF can be used to prove ISS if all $c_i > 0$ or all $d_i > 0$.

Improved modification method



Making discrete dynamics ISS

- Define $I_d := \{i \in \{1, \dots, n\} : d_i < 0\}$.
- $\forall i \in I_d$ restrict the frequency of jumps of Σ_i by

$$j - k \leq \delta_i(t - s) + N_0^i, \tag{ADT}$$

  where $\delta_i, N_0^i > 0$ and $(t, j), (s, k) \in \operatorname{dom} x$.
- It can be modeled by the clock

$$\begin{aligned} \dot{\tau}_i &\in [0, \delta_i], \quad \tau_i \in [0, N_0^i], \\ \tau_i^+ &= \tau_i - 1, \quad \tau_i \in [1, N_0^i]. \end{aligned}$$

- LF for $\tilde{\Sigma}_i$:

$$W_i(z_i) := \begin{cases} V_i(x_i), & i \notin I_d, \\ e^{L_i \tau_i} V_i(x_i), & i \in I_d. \end{cases}$$

Making discrete dynamics ISS

Proposition: ISS LF for modified subsystems

1) $\forall (z,u) \in \tilde{C}$ and $\forall y_i \in \tilde{f}_i(z,u)$,

$$W_i(z_i) \geq \max\left\{ \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u| \right\} \Rightarrow \dot{W}_i(z_i; y_i) \leq -\tilde{c}_i W_i(z_i),$$

where $\tilde{c}_i = c_i$ for $i \notin I_d$; $\tilde{c}_i = c_i - L_i \delta_i$ for $i \in I_d$ and

$$\begin{aligned} \tilde{\chi}_i &:= \chi_i, & \tilde{\chi}_{ij} &:= \chi_{ij}, & i &\notin I_d, \\ \tilde{\chi}_i &:= e^{L_i N_0^i} \chi_i, & \tilde{\chi}_{ij} &:= e^{L_i N_0^i} \chi_{ij}, & i &\in I_d. \end{aligned}$$

2) $\forall (z,u) \in \tilde{D}$ and $\forall y_i \in \tilde{g}_i(z,u)$,

$$W_i(y_i) \leq \max\left\{ e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u| \right\},$$

where $\tilde{d}_i = d_i$ for $i \notin I_d$ and $\tilde{d}_i = d_i + L_i$ for $i \in I_d$.

Example

Let $c_1 > 0$, $d_1 < 0$, $c_2 > 0$ and $d_2 < 0$.

- Instabilities are matched $\Rightarrow$ no modification.
- SGT $\Rightarrow$ LF V for interconnection with $c > 0$ and $d < 0$.

Let $c_1 > 0$, $d_1 < 0$, $c_2 < 0$ and $d_2 > 0$.

- Instabilities are not matched $\Rightarrow$ modify Σ_2.

$$\tilde{\Gamma} = \begin{bmatrix} 0 & \tilde{\chi}_{12} \\ \tilde{\chi}_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{L_1 N_0^1}\chi_{12} \\ \chi_{21} & 0 \end{bmatrix}.$$

- SGT $\Rightarrow$

$$\chi_{12}\chi_{21} < e^{-L_1 N_0^1}.$$

- Since $L_1 = -d_1 + \varepsilon; \quad N_0^1 \geq 1 \Rightarrow$

$$\chi_{12}\chi_{21} \leq e^{d_1}.$$

Summary and Outlook

Main results

- New small-gain theorem for hybrid systems.
- If instabilities are matched $\Rightarrow$ no modification is needed.
- Less restrictive modification method

Outlook

- Extension to nonlinear χ_{ij} , using ideas from
  S. Dashkovskiy and A.M. Input-to-State Stability of Nonlinear Impulsive Systems, SICON, 2013.

Summary and Outlook

Main results

- New small-gain theorem for hybrid systems.
- If instabilities are matched $\Rightarrow$ no modification is needed.
- Less restrictive modification method

Outlook

- Extension to nonlinear χ_{ij} , using ideas from
  S. Dashkovskiy and A.M. Input-to-State Stability of Nonlinear Impulsive Systems, SICON, 2013.
- Can one avoid modifications if the instabilities are not matched?

Summary and Outlook

Main results

- New small-gain theorem for hybrid systems.
- If instabilities are matched $\Rightarrow$ no modification is needed.
- Less restrictive modification method

Outlook

- Extension to nonlinear χ_{ij} , using ideas from
  S. Dashkovskiy and A.M. Input-to-State Stability of Nonlinear Impulsive Systems, SICON, 2013.
- Can one avoid modifications if the instabilities are not matched?

Thank you for attention!